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On the Complete 2-Normed Stabilization of Cubic and Quadratic Functional Equations

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Abstract

In this article, the stabilization of following cubic and quadratic functional equations is studied in complete 2-normed space for a mapping h from a normed linear space into a complete 2-normed space.

Key words : Complete 2-normed space, Cubic and Quadratic functional equations.

1. Introduction

The question of stability of functional equations was triggered by a famous mathematician S. M. Ulam[12] in 1940. But the first solution of that question was given by D. H. Hyers[6] in 1941 who determined the stability of Cauchy functional equations in Banach spaces. After some time, the generalized version of the Hyers problem was established by T.M. Rassias in 1978 [13]. Further, by replacing the bound $\lambda(\|r\|^p + \|s\|^p)$ with general condition $\psi(r, s)$, the stability of functional equations was modified by P. Gavruta[9]. In 2002 Chang *et. al*.[8] examined the Hyers-Ulam stability of the following quadratic functional equation

$$h(2r-s)+h(2r+s)-h(r-s)-h(r+s)-6h(r)=0\,.$$

The functional equation

$$h(3r-s)+h(3r+s)-h(r-s)-h(r+s)-16h(r)=0$$

is also named as quadratic equation. The stability of Equation (2) was studied by Kenary *et.al.*[7] via fixed point approach. It was Park[14] who investigated the stabilization of functional equations in complete 2-normed spaces. In 2004, Park *et. al.*[14], established the stability of following cubic functional equations:

$$h(2r-s)+h(2r+s)-2h(r-s)-2h(r+s)-12h(r)=0$$

And $$h(3r-s)+h(3r+s)-3h(r-s)-3h(r+s)-48h(r)=0$$

Further, the stability of various functional equations was studied on various spaces such as RN-space, IRN-space, Orthogonal Space etc[1,2,3].

This article is divided in to four sections. Section 2 presents the basic review of some usual terminologies and results which are used in further sections. In Sections 3 and 4 we have examined the stability of quadratic and cubic functional equations in complete 2-normed spaces

2. Preliminaries :

It was Gahler[10, 11] who established the concept of 2-normed space and studied the following results:

Definition 1. Let us consider X be a vector space defined over the field R with $\dim X > 1$ and also let $\|.,.\| : X \times X \to R$ be the mapping which satisfies the following axioms for $r,s,t \in X$ and $\lambda \in R$:

(i) $\|r,\ s\| = 0 \Leftrightarrow$ *r* and *s* are linearly dependent,

(ii) $\|r,s\| = \|s,r\|$,

(iii) $\|\lambda r,s\| = |\lambda|\|r,s\|$,

(iv) $\|r,s+t\| \le \|r,\ s\| + \|r,\ t\|$,

Then, $\|.,.\|$ is known as a 2-norm defined on *X* and the relation $(X,\|.,.\|)$ is know as a 2-normed space.

Definition 2. [4, 5] A sequence $< r_1, r_2, ...r_n >$ is said to be a convergent if for $r \in X$ $\lim_{n\to\infty} \|s, r_n - r\| = 0$. Also we can write $\lim_{n\to\infty} r_n = r$.

Definition 3. [4, 5] A sequence $< r_1, r_2, ...r_n >$ is said to be a Cauchy sequence if for $s,t \in X$, where y and z are L.I. then we have

$$\lim_{i,m\to\infty} \|s, r_l - r_m\| = 0 \text{ and } \lim_{i,m\to\infty} \|t,\ r_l - r_m\| = 0.$$

Definition 4. [10] A complete 2-normed space is known as 2-Banach space.

Lemma 5. [11] For a sequence $< r_1, r_2, ...r_n >$ which is convergent in *X*, then the

$$\lim_{n\to\infty} \|s,\ r_n\| = \left\|s,\ \lim_{n\to\infty} r_n\right\|.$$

Lemma 6. [11] Suppose $(X,\|.,.\|)$ is a 2-normed space and $\|r,\ s\| = 0$ for all $s \in X$, then also $r = 0$ for each

$s \in X$.

3. Complete 2-Normed stabilization of quadratic equations :

In this section we deals with the complete 2-normed stability of following quadratic equations

$$h(2r-s)+h(2r+s)-h(r-s)-h(r+s)-6h(r)=0 \tag{1}$$

$$h(3r-s)+h(3r+s)-h(r-s)-h(r+s)-16h(r)=0 \tag{2}$$

Theorem 1. Let $h: X \to Y$ be a mapping, where X is a normed linear space and Y is compete 2-normed space with $0 \le \lambda < \infty$ and $0 < p < 2$ which satisfies the relation

$$\|h(2r+s)+h(2r-s)-h(r+s)-h(r-s)-6h(r),\ t\| \le \lambda(\|r\|^p+\|s\|^p) \tag{3}$$

Then, $R: X \to Y$ will be a unique quadratic function such that

$$\|R(r)-h(r),\ s\| \le \frac{4^{(l+1)(m-l)}-2^{lp(m-l)}}{(4-2^p)2}\lambda\|r\|^p,\ \forall\ r \in X \text{ and } s \in Y\ . \tag{4}$$

Proof : Taking $s = 0$ in the relation (3), we obtain

$$\|8h(r)-2h(2r),\ t\| \le \lambda\|r\|^p,$$

$$\left\|\frac{h(2r)}{4}-h(r),\ t\right\| \le \frac{1}{8}\lambda\|r\|^p. \tag{5}$$

Now, putting *r=2r* and then dividing throughout by 4 in (5), we get

$$\left\|\frac{h(2^2 r)}{4^2}-\frac{h(2r)}{4},\ t\right\| \le \frac{1}{2.4^2}\lambda\|2t\|^p \tag{6}$$

Again, taking $r = 2^i r$ and then dividing throughout by 4^i in (5), we get

$$\left\|\frac{h(2^{i+1} r)}{4^{i+1}}-\frac{h(2^i r)}{4^i},\ t\right\| \le \frac{1}{2.4^{i+1}}\lambda\|2^i r\|^p$$

$$\le \frac{1}{2.4^{i+1}}2^{ip}\lambda\|r\|^p \tag{7}$$

Now, we prove the sequence $< h(2^n r)/4^n >$ is convergent, so, let us consider *l* and *m* be two positive integers such that *m*>*l*, then, we have

$$\left\|\frac{h(2^l r)}{4^l}-\frac{h(2^m r)}{4^m},\ t\right\| \le \frac{1}{2}\sum_{i=l}^{m-1}\frac{2^{ip}}{4^{i+1}}\lambda\|r\|^p \tag{8}$$

As *m*, *l* tends to infinity, we obtain

$$\lim_{m,l\to\infty}\left\|\frac{h(2^l r)}{4^l}-\frac{h(2^m r)}{4^m},\ t\right\|=0$$

Thus, the sequence $< h(2^n r)/4^n >$ is assumed as Cauchy sequence and hence convergent because Y is a complete 2-normed space. Now, let us define a map $R : X \to Y$ as

$$\lim_{n\to\infty}\frac{h(2^n r)}{4^n}=R(r)$$

which satisfies (1), thus, we have

$$\|R(2r-s)+R(2r+s)-R(r-s)-6R(r)-R(r+s),\quad t\|$$

$$=\lim_{i\to\infty}\frac{1}{4^i}\left\|h(2^{i+1}r+2^i s)+h(2^{i+1}r-2^i s)-h(2^i r+2^i s)-h(2^i r-2^i s)-6h(2^i r),\ t\right\|$$

$$\le\lim_{i\to\infty}\frac{1}{4^i}\lambda(\|2^i r\|^p+\|2^i s\|^p)\ \le\lim_{i\to\infty}\frac{2^{ip}}{4^i}\lambda\|r\|^p+\lim_{i\to\infty}\frac{2^{ip}}{4^i}\|s\|^p=0$$

$$\|R(2r-s)+R(2r+s)-R(r-s)-6R(r)-R(r+s),\ t\|=0$$

To prove the required result (4) using (8), we get

$$\|R(r)-h(r),s\|=\lim_{m\to\infty}\left\|\frac{h(2^m r)}{4^m}-h(r),s\right\|,$$

$$\le\frac{4^{(l+1)(m-l)}-2^{lp(m-l)}}{2(4-2^p)}\lambda\|r\|^p.$$

For uniqueness of the map $R : X \to Y$, let us take $R^{'} : X \to Y$ as another quadratic system and using equation (1), we get

$$\|R(r)-R^{'}(r),s\|=\frac{1}{4^n}\|R(2^n r)-R^{'}(2^n r),s\|,$$

$$\le\frac{1}{4^n}(\|R(2^n r)-h(2^n r),s\|+\|R^{'}(2^n r)-h(2^n r),s\|)$$

As $n\to\infty$ right hand side approaches to zero, thus we get $R(r)=R^{'}(r)$ for all $r\in X$. Hence the map $R : X \to Y$ is unique. Hence proved.

Theorem 2. Let $h : X \to Y$ be a mapping, where X is a normed linear space and Y is a complete 2-normed space with $0\le\lambda<\infty$ and $p>2$ which satisfies

$$\|h(2r-s)+h(2r+s)-h(r-s)-h(r+s)-6h(r),\quad t\|\le\lambda(\|r\|^p+\|s\|^p) \tag{9}$$

Then, $R : X \to Y$ will be a unique quadratic function such that

$$\|h(r)-R(r),s\| \le \frac{2^{lp(m-l)}-4^{(l)(m-l)}}{2(2^p-4)}\lambda\|r\|^p \tag{10}$$

Proof: Substituting $y = 0$ in the inequality (9), we obtain

$$\|2h(2r)-8h(r),\ t\| \le \lambda\|r\|^p$$

$$\left\|4h(\frac{r}{2})-h(r),\ t\right\| \le \frac{1}{2}\lambda\left\|\frac{r}{2}\right\|^p . \tag{11}$$

Taking $r/2 = r$ and then multiplying throughout by 4 in (11), we obtain

$$\left\|4^2h(\frac{r}{2^2})-4h(\frac{r}{2}),\ t\right\| \le \frac{1}{2}4\lambda\left\|\frac{r}{2^2}\right\|^p ,$$

Again taking $r = r/2^i$ and then multiplying throughout with in (11), we get

$$\left\|4^{i+1}h(\frac{r}{2^{i+1}})-4^ih(\frac{r}{2^i}),\ t\right\| \le \frac{1}{2}4^i\lambda\left\|\frac{r}{2^{i+1}}\right\|^p$$

To prove the sequence $< 4^n h(r/2^n) >$ is convergent. So, let m and l be two positive integers such that $m>l$, then

$$\left\|4^lh(\frac{r}{2^l})-4^mh(\frac{r}{2^m}),\ t\right\| \le \frac{\lambda}{2}\sum_{i=l}^{m-1}\frac{4^i}{2^{ip}}\|r\|^p \tag{12}$$

Limiting on both sides of (12), we found

$$\lim_{m,l\to\infty}\left\|4^lh(\frac{r}{2^l})-4^mh(\frac{r}{2^m}),\ t\right\| = 0$$

Thus, the sequence $< h(2^n r)/4^n >$ is assumed as Cauchy sequence and hence convergent also because Y is a complete 2-normed space. Now, let us define a map $R : X \to Y$ and the remaining part can be proved on the same lines of Theorem 1.

Theorem 3. Let $h : X \to Y$ be a mapping, where X is a normed linear space and Y is a complete 2-normed space with $0 \le \lambda < \infty$ and $0 < p < 2$ which satisfies

$$\|h(3r-s)+h(3r+s)-h(r-s)-h(r+s)-16h(r),\ t\| \le \lambda(\|r\|^p+\|s\|^p) \tag{13}$$

Then, $R : X \to Y$ will be a unique quadratic function such that

$$\|R(r)-h(r),\ s\| \le \frac{9^{(l+1)(m-l)}-3^{lp(m-l)}}{2(9-3^p)}\lambda\|r\|^p \tag{14}$$

Proof: Substituting $s = 0$ in the inequality (13), we obtain

$$\|18h(r) - 2h(3r),\ t\| \le \lambda \|r\|^p$$

$$\left\| h(r) - \frac{h(3r)}{9}, t \right\| \le \frac{1}{18} \lambda \|r\|^p \tag{15}$$

Taking $3r = r$ and then dividing throughout by 9 in (15), we obtain

$$\left\| \frac{h(3r)}{9} - \frac{h(3^2 r)}{9^2},\ t \right\| \le \frac{1}{2.9^2} \lambda \|3r\|^p \tag{16}$$

Again taking $3^i r$ $= r$ and then dividing throughout by 9^i in (15), we obtain

$$\left\| \frac{h(3^{i+1} r)}{9^{i+1}} - \frac{h(3^i r)}{9^i},\ t \right\| \le \frac{1}{2.9^{i+1}} \lambda \|3^i r\|^p,$$

$$\le \frac{1}{2.9^{i+1}} 3^{ip} \lambda \|r\|^p. \tag{17}$$

To prove the sequence $< h(3^n r)/9^n >$ is convergent. Therefore, let m, l be positive integers such that $m>l$, then

$$\left\| \frac{h(3^l r)}{9^l} - \frac{h(3^m r)}{9^m},\ t \right\| \le \frac{\lambda}{2} \sum_{i=l}^{m-1} \frac{3^{ip}}{9^{i+1}} \lambda \|r\|^p \tag{18}$$

Limiting on both sides of (18), we found

$$\lim_{m,l\to\infty} \left\| \frac{h(3^l r)}{9^l} - \frac{h(3^m r)}{9^m},\ t \right\| = 0$$

Thus, the sequence $< h(3^n r)/9^n >$ is assumed as Cauchy sequence and hence convergent also because Y is a complete 2-normed space. Now, let us define a map $R : X \to Y$ as

$$\lim_{n\to\infty} \frac{h(3^n r)}{9^n} = R(x)$$

Which satisfies the equations (2), thus we have

$$\|R(3r - s) + R(3r + s) - R(r - s) - R(r + s) - 16R(r),\ t\|$$

$$= \lim_{i\to\infty} \frac{1}{9^i} \|h(3^{i+1} r - 3^i s) + h(3^{i+1} r + 3^i s) - h(3^i r - 3^i s) - h(3^i r + 3^i s) - 16h(3^i r),\ t\|$$

$$\le \lim_{i\to\infty} \frac{1}{9^i} \lambda (\|3^i r\|^p + \|3^i s\|^p)$$

$$\le \lim_{i\to\infty}\frac{3^{ip}}{9^{i}}\lambda\|r\|^{p} + \lim_{i\to\infty}\frac{3^{ip}}{9^{i}}\|s\|^{p} = 0$$

$$\|R(3r-s)+R(3r+s)-R(r-s)-R(r+s)-16R(r),\ t\| = 0$$

To prove the required result (14) using (18), we get

$$\|R(r)-h(r),\ s\| = \lim_{m\to\infty}\left\|\frac{h(3^{m}r)}{9^{m}}-h(r),\ s\right\|$$

$$\le \frac{9^{(l+1)(m-l)}-3^{lp(m-l)}}{2(9-3^{p})}\lambda\|r\|^{p}$$

For uniqueness of the map $R: X \to Y$, let us take $R^{'}: X \to Y$ as another quadratic system and using equation (2), we get

$$\|R(r)-R^{'}(r),s\| = \frac{1}{9^{n}}\|R(3^{n}r)-R^{'}(3^{n}r),s\|$$

$$\le \frac{1}{9^{n}}(\|R(3^{n}r)-h(3^{n}r),s\|+\|R^{'}(3^{n}r)-h(3^{n}r),s\|)$$

As $n \to \infty$ right hand side approaches to zero, thus we get $R(r)=R^{'}(r)$ for all $r \in X$. Hence the map $R: X \to Y$ is unique. Hence proved.

Theorem 3. Let $h: X \to Y$ be a mapping, where X is a normed linear space and Y is a complete 2-normed space with $0 \le \lambda < \infty$ and $p > 2$ which satisfies

$$\|h(3r-s)+h(3r+s)-h(r-s)-h(r+s)-16h(r),\ t\| \le \lambda(\|r\|^{p}+\|s\|^{p}) \tag{19}$$

Then, $R: X \to Y$ will be a unique quadratic function such that

$$\|R(r)-h(r),\ s\| \le \frac{3^{lp(m-l)}-9^{(l)(m-l)}}{2(3^{p}-9)}\lambda\|r\|^{p}. \tag{20}$$

Proof: Proof is similar to the above result.

4. Complete 2-Normed stabilization of cubic equations :

This section studies the complete 2-normed stability of the following cubic functional equations

$$h(2r-s)+h(2r+s)-2h(r-s)-2h(r+s)-12h(r) = 0 \tag{21}$$

$$h(3r-s)+h(3r+s)-3h(r-s)-3h(r+s)-48h(r) = 0 \tag{22}$$

Theorem 4. Let $h: X \to Y$ be a mapping, where X is a normed linear space and Y is a complete 2-normed space with $0 \le \lambda < \infty$ and $0 < p < 3$ which satisfies

$$\|h(2r-s)+h(2r+s)-2h(r-s)-2h(r+s)-12h(r),\ t\| \leq \lambda(\|r\|^p+\|s\|^p) \tag{23}$$

Then, $S : X \to Y$ will be a unique cubic function such that

$$\|h(r)-S(r),s\| \leq \frac{8^{(l+1)(m-l)}-2^{lp(m-l)}}{2(8-2^p)}\lambda\|r\|^p \tag{24}$$

Proof: Substituting $s = 0$ in the inequality (23), we obtain

$$\|16h(r)-2h(2r),\ t\| \leq \lambda\|r\|^p$$

$$\left\|h(r)-\frac{h(2r)}{8},\ t\right\| \leq \frac{1}{16}\lambda\|r\|^p \tag{25}$$

Taking $2r = r$ and then dividing throughout by 8 in (25), we obtain

$$\left\|\frac{h(2r)}{8}-\frac{h(2^2r)}{8^2},\ t\right\| \leq \frac{1}{2.8^2}\lambda\|2r\|^p \tag{26}$$

Again taking $2^i r = r$ and then dividing throughout by 8^i in (25), we obtain

$$\begin{aligned}\left\|\frac{h(2^{i+1}r)}{8^{i+1}}-\frac{h(2^ir)}{8^i},\ t\right\| &\leq \frac{1}{2.8^{i+1}}\lambda\|2^ir\|^p, \\ &\leq \frac{1}{2.8^{i+1}}2^{ip}\lambda\|r\|^p.\end{aligned} \tag{27}$$

To prove the sequence $< h(2^n r)/8^n >$ is convergent. Therefore, let *m*, *l* be positive integers such that *m*>*l*, then

$$\left\|\frac{h(2^lr)}{8^l}-\frac{h(2^mr)}{8^m},\ t\right\| \leq \frac{\lambda}{2}\sum_{i=l}^{m-1}\frac{2^{ip}}{8^{i+1}}\lambda\|r\|^p \tag{28}$$

Limiting on both sides of (28), we found

$$\lim_{m,l\to\infty}\left\|\frac{h(2^lr)}{8^l}-\frac{h(2^mr)}{8^m},\ t\right\| = 0$$

Thus, the sequence $< h(2^n r)/8^n >$ is assumed as Cauchy sequence and hence convergent also because Y is a complete 2-normed space. Now, let us define a cubic map $S : X \to Y$ as

$$\lim_{n\to\infty}\frac{h(2^nr)}{8^n} = S(r)$$

which satisfies the equations (21), thus we have

$$\|S(2r-s)+S(2r+s)-2S(r-s)-2S(r+s)-12R(r),\ t\|$$

$$=\lim_{i\to\infty}\frac{1}{8^i}\|h(2^{i+1}r+2^i s)+h(2^{i+1}r-2^i s)-2h(2^i r+2^i s)-2h(2^i r-2^i s)-12h(2^i r),\ t\|$$

$$\le \lim_{i\to\infty}\frac{1}{8^i}\lambda(\|2^i r\|^p+\|2^i s\|^p)\le \lim_{i\to\infty}\frac{2^{ip}}{8^i}\lambda\|r\|^p+\lim_{i\to\infty}\frac{2^{ip}}{8^i}\lambda\|s\|^p=0$$

$$\|S(2r-s)+S(2r+s)-2S(r-s)-2S(r+s)-12R(r),\ t\|=0$$

To prove the required result (24) using (28), we get

$$\|S(r)-h(r),\ s\|=\lim_{m\to\infty}\left\|\frac{h(2^m r)}{8^m}-h(r),\ s\right\|\le\frac{8^{(l+1)(m-l)}-2^{lp(m-l)}}{2(8-2^p)}\lambda\|r\|^p$$

For uniqueness of the map $S:X\to Y$, let us take $S^{'}:X\to Y$ as another cubic system and using equation (21), we get

$$\|S(r)-S^{'}(r),\ s\|=\frac{1}{8^n}\|S(2^n r)-S^{'}(2^n r),\ s\|$$

$$\le\frac{1}{8^n}(\|S(2^n r)-h(2^n r),\ s\|+\|S^{'}(2^n r)-h(2^n r),s\|)$$

As $n\to\infty$ right hand side approaches to zero, thus we get $S(r)=S^{'}(r)$ for all $r\in X$. Hence the map $S:X\to Y$ is unique. Hence proved.

Theorem 5. Let $h:X\to Y$ be a mapping, where X is a normed linear space and Y is a complete 2-normed space with $0\le\lambda<\infty$ and $p>3$ which satisfies

$$\|h(2r-s)+h(2r+s)-2h(r-s)-2h(r+s)-12h(r),\ t\|\le\lambda(\|r\|^p+\|s\|^p)\tag{29}$$

Then, $S:X\to Y$ will be a unique cubic function such that

$$\|S(r)-h(r),\ s\|\le\frac{2^{lp(m-l)}-8^{(l)(m-l)}}{2(2^p-8)}\lambda\|r\|^p.\tag{30}$$

Proof: Substituting $s = 0$ in the inequality (29), we obtain

$$\|16h(r)-2h(2r),\ t\|\le\lambda\|r\|^p$$

$$\left\|8h(\frac{r}{2})-h(r),\ t\right\|\le\frac{1}{2}\lambda\left\|\frac{r}{2}\right\|^p\tag{31}$$

Taking $r/2=r$ and then multiplying throughout by 8 in (31), we obtain

$$\left\| 8^2 h(\frac{r}{2^2}) - 8h(\frac{r}{2}),\ t \right\| \leq \frac{1}{2} 8\lambda \left\| \frac{r}{2^2} \right\|^p$$

Again taking $r/2^i = r$ and then multiplying throughout by 8^i in (31), we obtain

$$\left\| 8^{i+1} h(\frac{r}{2^{i+1}}) - 8^i h(\frac{r}{2^i}),\ t \right\| \leq \frac{1}{2} 8^i \lambda \left\| \frac{r}{2^{i+1}} \right\|^p$$

To prove the sequence $< 8^n h(x/2^n) >$ is convergent. Therefore, let *m, l* be positive integers such that *m*>*l*, then

$$\left\| 8^l h(\frac{r}{2^l}) - 8^m h(\frac{r}{2^m}),\ t \right\| \leq \frac{\lambda}{2} \sum_{i=l}^{m-1} \frac{8^i}{2^{ip}} \|r\|^p \tag{32}$$

Limiting on both sides of (32), we get

$$\lim_{m,l\to\infty} \left\| 8^l h(\frac{r}{2^l}) - 8^m h(\frac{r}{2^m}),\ t \right\| = 0$$

Thus, the sequence $< 8^n h(x/2^n) >$ is assumed as Cauchy sequence and hence convergent also because Y is a complete 2-normed space. Now, let us define a cubic map $S : X \to Y$ as

$$\lim_{n\to\infty} 8^n h(\frac{r}{2^n}) = S(r) \text{ for all } r \in X .$$

The remaining part is similar to Theorem 4.

Theorem 6. Let $h : X \to Y$ be a mapping, where X is a normed linear space and Y is a complete 2-normed space with $0 \leq \lambda < \infty$ and $0 < p < 3$ which satisfies

$$\| h(3r-s) + h(3r+s) - 3h(r-s) - 3h(r+s) - 48h(r),\ t \| \leq \lambda(\|r\|^p + \|s\|^p) \tag{33}$$

Then, $S : X \to Y$ will be a unique cubic function such that

$$\| S(r) - h(r),\ s \| \leq \frac{27^{(l+1)(m-l)} - 3^{lp(m-l)}}{2(27 - 3^p)} \lambda \|r\|^p . \tag{34}$$

Proof: Substituting *s* = 0 in the inequality (33), we obtain

$$\| 2h(3r) - 54h(r),\ t \| \leq \lambda \|r\|^p$$

$$\left\| h(r) - \frac{h(3r)}{27},\ t \right\| \leq \frac{1}{54} \lambda \|r\|^p \tag{35}$$

Taking 3*r* = *r* and then dividing throughout by 27 in (35), we obtain

$$\left\|\frac{h(3r)}{27}-\frac{h(3^2 r)}{27^2},\ t\right\| \le \frac{1}{2.27^2}\lambda\|3r\|^p \tag{36}$$

Again taking $3^i r$ = *r* and then dividing throughout by 27^i in (35), we obtain

$$\begin{aligned}\left\|\frac{h(3^i r)}{27^i}-\frac{h(3^{i+1} r)}{27^{i+1}},\ t\right\| &\le \frac{1}{2.27^{i+1}}\lambda\|3^i r\|^p, \\ &\le \frac{1}{2.27^{i+1}}3^{ip}\lambda\|r\|^p.\end{aligned} \tag{37}$$

To prove the sequence $< h(3^n r)/27^n >$ is convergent. Therefore, let *m, l* be positive integers such that *m*>*l*, then

$$\left\|\frac{h(3^l r)}{27^l}-\frac{h(3^m r)}{27^m},\ t\right\| \le \frac{\lambda}{2}\sum_{i=l}^{m-1}\frac{3^{ip}}{27^{i+1}}\lambda\|r\|^p \tag{38}$$

Limiting on both sides of (38), we get

$$\lim_{m,l\to\infty}\left\|\frac{h(3^l r)}{27^l}-\frac{h(3^m r)}{27^m},\ t\right\| = 0$$

Thus, the sequence $< h(3^n r)/27^n >$ is assumed as Cauchy sequence and hence convergent also because Y is a complete 2-normed space. Now, let us define a cubic map $S : X \to Y$ as

$$\lim_{n\to\infty}\frac{h(3^n r)}{27^n} = S(r) \text{ for all } r \in X.$$

which satisfies the equations (22), thus we have

$$\|S(3r-s)+S(3r+s)-3S(r-s)-3S(r+s)-48S(r),\ t\|$$

$$= \lim_{i\to\infty}\frac{1}{27^i}\|h(3^{i+1}r-3^i s)+h(3^{i+1}r+3^i s)-3h(3^i r-3^i s)-3h(3^i r+3^i s)-48h(3^i r),\ t\|$$

$$\le \lim_{i\to\infty}\frac{1}{27^i}\lambda(\|3^i r\|^p+\|3^i s\|^p) \le \lim_{i\to\infty}\frac{3^{ip}}{27^i}\lambda\|r\|^p + \lim_{i\to\infty}\frac{3^{ip}}{27^i}\lambda\|s\|^p = 0$$

$$\|S(3r-s)+S(3r+s)-3S(r-s)-3S(r+s)-48S(r),\ t\| = 0$$

To prove the required result (34) using (38), we get

$$\begin{aligned}\|h(r)-S(r), s\| &= \lim_{m\to\infty}\left\|h(r)-\frac{h(3^m r)}{27^m}, s\right\| \\ &\le \frac{27^{(l+1)(m-l)}-3^{lp(m-l)}}{2(27-3^p)}\lambda\|r\|^p\end{aligned}$$

For uniqueness of the map $S : X \to Y$, let us take $S' : X \to Y$ as another cubic system and using equation (22), we get

$$\begin{aligned}\left\|S(r)-S^{'}(r),\ s\right\| &= \frac{1}{27^n}\left\|S(3^n r)-S^{'}(3^n r),\ s\right\| \\ &\le \frac{1}{27^n}\left(\left\|S(3^n r)-h(3^n r),\ s\right\|+\left\|S^{'}(3^n r)-h(3^n r),s\right\|\right)\end{aligned}$$

As $n \to \infty$ right hand side approaches to zero, thus we get $S(r)=S^{'}(r)$ for all $r \in X$. Hence the map $S : X \to Y$ is unique. Hence proved.

Theorem 7. Let $h : X \to Y$ be a mapping, where X is a normed linear space and Y is a complete 2-normed space with $0 \le \lambda < \infty$ and $p > 3$ which satisfies

$$\left\|h(3r-s)+h(3r+s)-3h(r-s)-3h(r+s)-48h(r),\ t\right\| \le \lambda\left(\left\|r\right\|^p+\left\|s\right\|^p\right) \tag{39}$$

Then, $S : X \to Y$ will be a unique cubic function such that

$$\left\|h(r)-S(r),s\right\| \le \frac{3^{lp(m-l)}-27^{(l)(m-l)}}{(3^p-27)}\lambda\left\|r\right\|^p \tag{40}$$

Proof: Similar to Theorem 6.

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