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A Generalized Fixed Point Theorem in G-Metric Space

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Abstract

In this paper, we consider the concept of non compatible mapping as property (E.A.) in G-metric space and prove some common fixed point theorems.

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Key words : Common fixed point, Property (E.A.), G-metric space.

1 Introduction

The Banach fixed point theorem for contraction mapping has been generalized and extended in many directions. Amari and Moutawakil¹ in 2002 introduced a generalization of non compatible mappings as property (E.A.) in metric spaces. Pathak *et al.* in 2010 used the property (E.A.) in metric spaces to prove common fixed point results. Mustafa and Sims in 2004 introduce the concept of G-metric space. Saadati *et al.* in 2010 extended the concept of w-distance in metric space to G-metric space. Chaudhary *et al.* in 2012 introduce the concept of compatibility and weak compatibility of mappings in G-metric spaces.

The aim of this paper is to obtain a generalized common fixed point theorem for non compatible mappings in G-metric space.

2 Preliminaries:

Definition : Let X be a non empty set and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following:

- (i) $G(x, y, z) = 0$ if $x = y = z$,
- (ii) $G(x, x, y) > 0$, for all $x, y \in X$, with $x \neq y$,
- (iii) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),
 (v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality),
 then the function is called a generalized metric *i.e.* G-metric on X and the pair (X, G) is a G-metric space.

Definition : Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence of points in X . Then $\{x_n\}$ is G-convergent to x if $\lim_{n \rightarrow \infty} G(x, x_n, x_m) = 0$, *i.e.*, for each $\epsilon > 0$ there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq N$. We call that x is the limit of sequence and we write $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$.

Definition : Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is said to be a G-cauchy sequence for each $\epsilon > 0$ there exist a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $l, m, n \geq N$.

Definition : A G-metric space (X, G) is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition : Let f and g be two self mappings on a metric space (X, d) . The mappings f and g are said to be compatible if $\lim_{n \rightarrow \infty} d(fg x_n, g f x_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = z$ for some $z \in X$.

Definition : Let (X, G) be a G-metric space and $T : X \rightarrow X$ be a self mapping on (X, G) . Then T is said to be a contraction if

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z), \text{ for all } x, y, z \in X \text{ where } 0 \leq \alpha < 1.$$

Definition : Let f and g be two self mappings on a G-metric space (X, G) . The mappings f and g are said to be compatible if $\lim_{n \rightarrow \infty} G(fg x_n, g f x_n, g f x_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = z$ for some $z \in X$.

Definition : Two mappings f and g are said to be weakly compatible if they commute at coincident points.

Definition : Let f and g be two self mappings on a metric space (X, d) . The pair (f, g) is said to satisfy property (E. A.) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = z$ for some $z \in X$.

3 Main Theorem :

Theorem : Let (X, G) be a complete G-metric space and f, g be two self mappings on (X, G) satisfies the following conditions:

i. f and g satisfy property (E.A.), (3.1)

ii. $g(X)$ is closed subspace of X , (3.2)

iii. $G(fx, fy, fz) \leq \alpha G(fx, gy, gz) + \beta G(gx, fy, gz) + \gamma G(gx, gy, fz) + \delta G(fx, fy, gz) + \zeta G(fx, gy, fz)$ (3.3)

for every $x, y, z \in X$ and $\alpha, \beta, \gamma, \delta, \zeta \geq 0$ with $0 \leq \alpha + 3\beta + 3\gamma + 3\delta + 3\zeta < 1$. Then f and g have a unique common fixed point in X provided f and g are weakly compatible self maps.

Proof : Since f and g satisfy property (E. A.), therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = u \in X$. Since $g(X)$ is a closed subspace of X , so every convergent sequence of points of $g(X)$ has a limit point in $g(X)$. Thus $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = u$ for some $u \in X$. This shows that $u = ga$ for some $a \in X$.

From (3.3), we have

$$G(fa, f x_n, f x_n) \leq \alpha G(fa, g x_n, g x_n) + \beta G(ga, f x_n, g x_n) + \gamma G(ga, g x_n, f x_n) + \delta G(fa, f x_n, g x_n) + \zeta G(fa, g x_n, f x_n)$$

Letting $n \rightarrow \infty$ and using $0 \leq \alpha + 3\beta + 3\gamma + 3\delta + 3\zeta < 1$, we have $u = fa$. Thus $u = fa = ga$ *i.e.* a is the coincidence point of f and g .

Since f and g are weakly compatible, therefore $fu = gu = fga = gfa$.

Again from (3.3)

$$G(fu, fa, fa) \leq \alpha G(fu, ga, ga) + \beta G(gu, fa, ga) + \gamma G(gu, ga, fa) + \delta G(fu, fa, ga) + \zeta G(fu, ga, fa)$$

It implies that $fu = u$. Hence $fu = gu = u$ i. e. u is common fixed point of f and g .

For uniqueness, we consider that $v \neq u$ be another common fixed point of f and g .

From (3.3), we have

$$\begin{aligned} G(u, v, v) &= G(fu, fv, fv) \\ &\leq \alpha G(fu, gv, gv) + \beta G(gu, fv, gv) + \gamma G(gu, gv, fv) + \delta G(fu, fv, gv) + \zeta G(fu, gv, fv) \\ &\leq (\alpha + \beta + \gamma + \delta + \zeta)G(u, v, v) \\ &< G(u, v, v) \end{aligned}$$

A contradiction, thus $u = v$. Hence the theorem.

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