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New Generalization of Homeomorphism in Topological Spaces

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Abstract

In this paper we introduce a new class of closed maps namely $g^{\#}s$-closed maps also introduce a new class of homeomorphisms called $g^{\#}s^*$-homeomorphisms and prove that the set of all $g^{\#}s^*$-homeomorphisms form a group under the operation composition of maps.

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1 Introduction

The notion homeomorphism plays an important role in topology. A homeomorphism is a bijective map $f: X \rightarrow Y$ when both f and f^{1} are continuous. Veera kumar[5] in 2002 introduced the concept of $g^{\#}$-semi-closed sets in topological spaces. In this paper we first introduce a new class of closed maps namely $g^{\#}s$-closed maps and then we introduce and study $g^{\#}s^*$-homeomorphisms in a topological space. We also prove that the set of all $g^{\#}s^*$-homeomorphisms form a group under the operation of composition of maps.

2 Preliminaries :

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation

axioms are assumed unless otherwise mentioned. For a subset A of space (X, τ) the cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A in X respectively.
We recall the following definitions:

Definition 2.01 : A subset A of a topological space (X, τ) is called semi-open [1] (resp. semi- closed[1]) if $A \subseteq cl(int(A))$ (resp. $int(cl(A)) \subseteq A$).

The semi-closure[3] of a subset A of X (denoted by scl(A)) is defined to be the intersection of all semi-closed sets containing A.

Definition 2.02 : A subset A of a topological space (X, τ) is called α-open [2] (resp. α-closed [2]) if $A \subseteq int(cl(int(A)))$ (resp. $cl(int(cl(A)))$).

The α-closure of a subset A of X (denoted by $\alpha cl(A)$) is defined to be the intersection of all α-closed sets containing A.

Definition 2.03: A subset A of a topological space (X, τ) is called

(i) αg-closed [4] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. The complement of αg-closed set is called αg-open.

(ii) $g^{\#}$s-closed [5] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open in X. The complement of $g^{\#}$s-closed set is called $g^{\#}$s-open.

Definition 2.04 : A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) $g^{\#}$s-continuous [5] if the inverse image of every σ-closed set in Y is $g^{\#}$s-closed in X.

(ii) $g^{\#}$s-irresolute [5] if the inverse image of every $g^{\#}$s-closed set in Y is $g^{\#}$s-closed in X.

3 $g^{\#}$s-Closed Maps :

In this section we introduce the following definitions.

Definition 3.01 : Let (X, τ) be a topological space and $A \subseteq X$. We define the $g^{\#}$s-closure of A (briefly $g^{\#}$s-cl(A)) is defined as the intersection of all $g^{\#}$s-closed sets containing A i.e. $g^{\#}s\text{-}cl(A) = \cap \{B : A \subseteq B \text{ and } B \in G^{\#}SC(X, \tau)\}$. Here $G^{\#}SC$ represent family of $g^{\#}$s-closed sets.

Theorem 3.02 : Let (X, τ) be a topological space and $A \subseteq X$ then following properties are follows :

(i) $g^{\#}$s-cl(A) is the smallest $g^{\#}$s-closed set containing A.

(ii) A is $g^{\#}$s-closed iff $g^{\#}s\text{-}cl(A) = A$.

Proof : Follows from definitions.

Theorem 3.03 : For any two subsets A and B of (X, τ)

(i) If $A \subseteq B$ then $g^{\#}s\text{-}cl(A) \subseteq g^{\#}s\text{-}cl(B)$.

(ii) $g^{\#}s\text{-}cl(A \cap B) \subseteq g^{\#}s\text{-}cl(A) \cap g^{\#}s\text{-}cl(B)$.

Proof : Immediately follows from definitions.

Theorem 3.04 : If $B \subseteq A \subseteq X$, B is a $g^{\#}$s-closed set relative to A and A is open and $g^{\#}$s-closed in (X, τ).

Then B is $g^{\#}$s-closed in (X, τ).

Corollary 3.05 : If A is $g^{\#}$s-closed set and B is closed set then $A \cap B$ is $g^{\#}$s-closed set.

Proof : Follows immediately since every closed set is *gs-closed.

Definition 3.06 : Let (X, τ) be a topological space and $A \subseteq X$ then We define $g^{\#}$s-interior of A (briefly $g^{\#}$s-int(A)) as the union of all $g^{\#}$s-open sets contained in A.

Lemma 3.07 : For any $A \subseteq X$, $\text{int}(A) \subseteq g^{\#}\text{s-int}(A) \subseteq A$.

Proof : Since every open set is $g^{\#}$s-open so proof straight forward .

Definition 3.08 : A map $f : (X, \tau) \to (Y, \sigma)$ is called $g^{\#}$s-closed (resp. $g^{\#}$s-open) if the image of every closed (resp. open) set in (X, τ) is $g^{\#}$s-closed (resp. $g^{\#}$s-open) in (Y, σ).

Theorem 3.09 : A map $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s-closed iff $g^{\#}\text{s-cl}(f(A)) \subseteq f(\text{cl}(A))$ for every subset A of (X, τ).

Proof : Follows from theorem (3.02) and (3.03).

Theorem 3.10 : A map $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s-closed iff for each subset A of (Y, σ) and for each open set U containing $f^{1}(A)$ there exists a $g^{\#}$s-open set V of (Y, σ) such that $A \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof : Let f is $g^{\#}$s-closed map. Let $A \subseteq Y$ and U be an open subset of (X, τ) such that $f^{-1}(A) \subseteq U$ then $V = (f(U^C))^C$ is a $g^{\#}$s-open set containing A such that $f^{-1}(V) \subseteq U$.

Conversely let A be a closed set in X then $f^{-1}((f(A))^C) \subseteq A^C$ and A^C is open in X. By assumption there exists a $g^{\#}$s-open set V of (Y, σ) s.t. $(f(A))^C \subseteq V$ and $f^{-1}(V) \subseteq A^C$ so $A \subseteq (f^{-1}(V))^C$. Hence $V^C \subseteq f(A) \subseteq f((f^{1}(V))^C) \subseteq V^C$ i.e. $f(A) = V^C$, since V^C is $g^{\#}$s-closed so f(A) is $g^{\#}$s-closed i.e. f is $g^{\#}$s-closed map.

Remark 3.11 : The following example shows that the composition of two $g^{\#}$s-closed maps need not be $g^{\#}$s-closed.

Example 3.12 : Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, Y\}$, $\eta = \{\phi, \{a\}, \{b, c\}, Z\}$. Define $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ by identity mapping then f and g both are $g^{\#}$s-closed map but their composition $gof : (X, \tau) \to (Z, \eta)$ is not a $g^{\#}$s-closed map.

Theorem 3.13 : Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \to (Z, \eta)$ be a $g^{\#}$s-closed map then their composition $gof : (X, \tau) \to (Z, \eta)$ is $g^{\#}$s-closed.

Proof : Follows from definitions.

Remark 3.14 : If $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s-closed and $g : (Y, \sigma) \to (Z, \eta)$ is closed then their composition $gof : (X, \tau) \to (Z, \eta)$ need not be a $g^{\#}$s-closed map as seen from the following example.

Example 3.15 : Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$, $\sigma = \{\phi, \{a\}, \{c\}, \{a, c\}, Y\}$, $\eta = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}, Z\}$. Define $f : (X, \tau) \to (Y, \sigma)$ byidentity mapping and $g : (Y, \sigma) \to (Z, \eta)$ by $g(a) = b$, $g(b) = a$, $g(c) = c$. Then f is $g^{\#}$s-closed map and g is closed map but their composition $gof : (X, \tau) \to (Z, \eta)$ is not a $g^{\#}$s-closed map.

Theorem 3.16 : Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two mappings such that their composition $gof : (X, \tau) \to (Z, \eta)$ be a $g^{\#}$s-closed map then the following statements are true.

(i) If f is continuous and serjective then g is $g^{\#}$s-closed map.

(ii) If g is $g^{\#}$s-irresolute and injective then f is $g^{\#}$s-closed map.

Theorem 3.17: Let f_A be the restriction of a map $f : (X, \tau) \to (Y, \sigma)$ to a subset A of (X, τ) then

(i) If $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s-closed and A is a closed subset of (X, τ), then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $g^{\#}$s-closed.

(ii) If $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s-closed (resp. closed) and $A = f^{-1}(B)$ for some closed (resp. $g^{\#}$s-closed) set B of (Y, σ) then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $g^{\#}$s-closed.

Proof : Proof is obvious.

Theorem 3.18 : For any bijective $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent.

(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $g^{\#}$s-continuous.
(ii) f is a $g^{\#}$s-open map and
(iii) f is a $g^{\#}$s-closed map.

Theorem 3.19: Let $f : (X, \tau) \to (Y, \sigma)$ be a $g^{\#}$s-open map then for a subset A of (X, τ), $f(\text{int}(A)) \subseteq g^{\#}s\text{-int}(f(A))$.

Theorem 3.20: A function $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s-open if and only if for any subset B of (Y, σ) and for any closed set A containing $f^{1}(B)$, there exists a $g^{\#}$s-closed set C of (Y, σ) containing B such that $f^{-1}(C) \subseteq A$.

Proof : Similar to theorem (2.10).

Corollary 3.21: A function $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s-open if and only if $f^{-1}(g^{\#}s\text{-cl}(A)) \subseteq \text{cl}(f^{-1}(A)$ for every subset A of (Y, σ).

Definition 3.22: A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a $g^{\#}$s*-closed (resp. $g^{\#}$s*-open) if the image f(A) is $g^{\#}$s-closed (resp. $g^{\#}$s-open) set in (Y, σ) for every $g^{\#}$s-closed (resp. $g^{\#}$s-open) set A in (X, τ).

Theorem 3.23: Every $g^{\#}$s*-closed map is $g^{\#}$s-closed map.

The converse is not true in general as it can be seen from the following example.

Example 3.24: Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by identity mapping then f is $g^{\#}$s-closed map but not $g^{\#}$s*-closed map.

Theorem 3.25: A map $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}$s*-closed iff $g^{\#}s\text{-cl}(f(A)) \subseteq f(g^{\#}s\text{-cl}(A))$ for every subset A of (X, τ).

Proof : Similar to theorem (3.09).

Theorem 3.26: For any bijection $f : (X, \tau) \to (Y, \sigma)$ the following are equivalent

(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $g^{\#}$s-irresolute,
(ii) f is a $g^{\#}$s*-open map and
(iii) f is a $g^{\#}$s*-closed map.

Proof : Similar to theorem (3.18).

4 $g^{\#}s^*$-Homeomorphisms :

In this section we introduce the following definitions.

Definition 4.01 : A bijection $f : (X, \tau) \to (Y, \sigma)$ is called $g^{\#}$s*-homeomorphism if both f and f^{-1} are $g^{\#}$s*-irresolute. We denote the family of all $g^{\#}$s*-homeomorphism of a topological space (X, τ) onto itself by $g^{\#}s^*\text{-h}(X, \tau)$.

Theorem 4.02 : If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $g^{\#}$s*-homeomorphism then their composition $gof : (X, \tau) \to (Z, \eta)$ is also $g^{\#}$s*-homeomorphism.

Proof : Let U be $g^{\#}s^*$-open set in (Z, η) then $g^{-1}(U)$ is $g^{\#}s^*$-open set in Y and so $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)$ is $g^{\#}s^*$-open in (X, τ) so gof is $g^{\#}s^*$-irresolute.

Again let V be $g^{\#}s^*$-open set in X then by hypothesis f(V) is $g^{\#}s^*$-open in Y and then $g(f(V)) = (gof)(V)$ is $g^{\#}s^*$-open in Z so $(gof)^{-1}$ is $g^{\#}s^*$-irresolute. Hence gof is a $g^{\#}s^*$-homeomorphism.

Theorem 4.03 : The set $g^{\#}s^*\text{-}h(X, \tau)$ is a group under the composition of maps.

Theorem 4.04 : Let $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $g^{\#}s^*$-homeomorphism then f induces an isomorphism from the group $g^{\#}s^*\text{-}h(X, \tau)$ onto the group $g^{\#}s^*\text{-}h(Y, \sigma)$.

Proof : Define $\theta_f : g^{\#}s^*\text{-}h(X, \tau) \rightarrow g^{\#}s^*\text{-}h(Y, \sigma)$ by $\theta_f(h) = fohof^{-1}$ for every $h \in g^{\#}s^*\text{-}h(X, \tau)$. Then θ_f is a bijection. Again for all $h_1, h_2 \in g^{\#}s^*\text{-}h(X, \tau)$, $\theta_f(h_1 oh_2) = fo(h_1 oh_2)of^{-1} = (foh_1of^{-1})o(foh_2of^{-1}) = \theta_f(h_1)o\theta_f(h_2)$ so θ_f is a homeomorphism and so it is an isomorphism induced by f.

Theorem 4.05 : $g^{\#}s^*$-homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof : Follows from theorem (4.02).

Theorem 4.06: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $g^{\#}s^*$-homeomorphism, then $g^{\#}s\text{-}cl(f^{-1}(A)) \subseteq f^{-1}(g^{\#}s\text{-}cl((B))$ for all $A \subseteq Y$.

Corollary 4.07: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $g^{\#}s^*$-homeomorphism, then $g^{\#}s\text{-}cl(f(A)) = f(g^{\#}s\text{-}cl((A))$ for all $A \subseteq X$.

Corollary 4.08: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $g^{\#}s^*$-homeomorphism, then $f(g^{\#}s\text{-}int(A)) = g^{\#}s\text{-}int(f(A))$ for all $A \subseteq X$.

Proof: Follows from corollary (4.07).

Corollary 4.09: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $g^{\#}s^*$-homeomorphism, then $f^{-1}(g^{\#}s\text{-}int(A)) = g^{\#}s\text{-}int(f^{1}(A))$ for all $A \subseteq Y$.

Proof: Follows from corollary (4.08).

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