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Product Fuzzy Syntopogenous Spaces and Fuzzy syntopogenous structure

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Abstract

The present paper deals with semi-topogenous order, Fuzzy-semi-topogenous order, perfect and biperfect fuzzy semi-topogenous order, inverse image of Fuzzy Semi-topogenous order and inverse image of fuzzy syntopogenous structure, continuity & some interesting results on product fuzzy syntopogenous spaces.

Key words : Fuzzy semi-topogenous order, perfect, biperfect, fuzzy semi-topogenous structure, continuity.

Introduction

L.A Zadeh was the first Mathematician who invented fuzzy set.

Csaszar gave a new method for foundation of general topology based on the theory of syntopogenous structures. Special cased of these structures are the topologies, the proximities and the uniformities. In the case of fuzzy structures, there are at least two notions of fuzzy uniformities, one of them is due to Hutton & the other due to Lowen. There are also two definitions of fuzzy proximities. The definition of fuzzy proximity given by Katsaras is closely connected with Hutton fuzzy uniformities. The other definition of a fuzzy proximity was given by Artico-Moresco & it is closely connected with Lowen Fuzzy uniformities.

We now study the fuzzy syntopogenous structure which agrees very well with fuzzy nhd structures, the Lowen fuzzy uniformities & the Artico-Moresco fuzzy proximities. There is a one to one correspondence between the fuzzy nhd structures & the so called perfect fuzzy topogenous structures. Also there is a one to one correspondence between the Artico-Morsco fuzzy proximities & the symmetrical fuzzy topogenuos structures.

In this paper we discuss inverse image of a fuzzy semi-topogenous order, inverse image of fuzzy syntopogenous structure, continuity, some results on product fuzzy syntopogenous spaces.

2. Semi Topogenous Order & Fuzzy Semi-Topogenous Order :

Definition 2.1 : An order relation << on the subsets of a set X is called a smitopogenous order on X if it satisfies the following conditions.

(a) $X << X$ & $\phi << \phi$

(b) $A << B$ implies that $A \subset B$

(c) $A_1 \subset A << B \subset B_1$ implies $A_1 << B_1$

The semi-topogenous order << is called topogenous if it satisfies also

(d_1) $A_i << B_i$, $i = 1, 2$ imply that

$A_1 \cup A_2 << B$

(d_2) $A << B_i, i = 1, 2$ imply that

$A << B_1 \cap B_2$

We now give the following definitions :

Definition 2.2 :

A fuzzy semi-topogenous order on X is a function,

$\zeta : I^x \text{ x } I^x \to 1$ which satisfies the following conditions.

(i) $\zeta(0,0) = \zeta(1,1) = 1$

(ii) $\zeta(\mu, \rho) \le [1 - \mu\ (x)] \vee \rho(x)$ for every $x \in X$.

(iii) $\mu_1 \le \mu$ & $\rho \le \rho_1$ imply that

$\zeta(\mu, \rho) \le \zeta(\mu_1, \rho_1)$

(iv) $|\zeta(\mu, \rho) - \zeta(\mu', \rho')|$

$\le \|\mu - \mu'\| + \|\rho - \rho'\|$

where for $\sigma_1, \sigma_2 \in I^x$,

$$\| \sigma_1 - \sigma_2 \| = \operatorname*{Sup}_{x \in X} | \sigma_1(x) - \sigma_2(x) |$$

The fuzz semi-topogenous order ζ is called topogenous if it satisfies also,

(v) $\zeta(\mu_1 \vee \mu_2, \rho)$

Definition 2.3 : A fuzzy semi-topogenous order ζ on X is called

(i) Perfect if $\zeta\left(\bigvee_{j\in J} \mu_j, \rho\right) = \bigwedge_{j\in J} \zeta(\mu_j, \rho)$

(ii) Biperfect if it is perfect & $\zeta\left(\mu \bigwedge_{j\in J} \rho_j\right) = \bigwedge_{j\in J} \zeta(\mu, \rho_j)$

(iii) Symmetrical if $\zeta = \zeta^C$

ζ is biperfect iff both $\zeta \,\&\, \zeta^C$ are perfect.

Definition 2.4 : If ζ_1, ζ_2 are fuzzy semi-topogenous order on X, then $\zeta = \zeta_1 \circ \zeta_2$ is defined by $\zeta(\mu, \rho) = \bigvee_{A \subset X} \zeta_2(\mu, A) \wedge \zeta_1(A, \rho)$

3. Inverse Image of a Fuzzy semi-Topogenous order :

Let $f : X \to Y$ be a function and let ζ be a fuzzy semi-topogenous order on Y. The Mapping.

$\zeta' : I^x \times I^x \to I, \zeta'(\mu, \rho) = \zeta(f(\mu), 1 - f(1 - \rho))$

is a fuzzy semi-topogenous order on X.

We will call ζ' the inverse image of ζ by the mapping f and we will denote if by $f^{-1}(\zeta)$.

Proposition 3.1 : Let f be a function from X to Y and let ζ, ζ' be fuzzy semi-topogenous orders on Y. Then :

(1) For $\mu, \rho \in I^Y$ we have

$$f^{-1}(\zeta)\left(f^{-1}(\mu), f^{-1}(\rho)\right) \geq \zeta(\mu, \rho) \qquad (*)$$

In case f is onto, we have equality in (*)

(2) $f^{-1}(\zeta)$ is the coarsest fuzzy semi-topogenous order ζ_1 on X for which

$$\zeta_1\left(f^{-1}(\mu), f^{-1}(\rho)\right) \geq \zeta(\mu, \rho)$$

for all $\mu, \rho \in = I^Y$

(3) If $\zeta \leq \zeta'$, then $f^{-1}(\zeta) \leq f^{-1}(\zeta')$. In case f is onto, the converse is also true.

(4) If $(\zeta_\lambda : \lambda \in \wedge)$ is a family of fuzzy semi-topogenous orders on Y, then

$$f^{-1}\left(\operatorname{Sup}_{\lambda} \zeta_\lambda\right) = \operatorname{Sup}_{\lambda} f^{-1}(\zeta_\lambda)$$

(5) $\left[f^{-1}(\zeta)\right]^q = f^{-1}(\zeta^q)$

(6) $\left[f^{-1}(\zeta)\right]^\rho = f^{-1}(\zeta^\rho)$ and $\left[f^{-1}(\zeta)\right]^b = f^{-1}(\zeta^b)$.

(7) if ζ is topogenous, then $f^{-1}(\zeta)$ is topogenous.

(8) if ζ is perfect (resp. biperfect), then $f^{-1}(\zeta)$ is perfect (resp. biperfect),

(9) $\left[f^{-1}(\zeta)\right]^C = f^{-1}(\zeta^C)$

(10) if ζ is symmetrical, then $f^{-1}(\zeta)$ is also symmetrical.

(11) $\zeta_1 = f^{-1}(\zeta), \zeta_2 = f^{-1}(\zeta')$ and $\zeta_3 = f^{-1}(\zeta o \zeta')$, then ζ_3 a coarser than $\zeta_1 o \zeta_2$.

If f is onto, then $\zeta_3 = \zeta_1 o \zeta_2$.

Proof :

(1) Since $f\left(f^{-1}(\mu)\right) \leq \mu$ and $f\left(1 - f^{-1}(\rho)\right) \leq 1 - \rho, (*)$ follows from the fact that ζ is a fuzzy semi-topogenous order. If f is onto then $f\left(f^{-1}(\mu)\right) = \mu$ and $f\left(1 - f^{-1}(\rho)\right) = 1 - \rho$ and so we have equality in (*).

(2) If follows from (1) and from the fact for $\mu, \rho \in I^X$ we have

$$f^{-1}\left(f(\mu)\right) \geq \mu \text{ and } f^{-1}(1 - f(1 - \rho)) \leq \rho.$$

(3) It follows from (1).

(4) By (3), we have

$$f^{-1}\left(\vee_{\lambda} \zeta_\lambda\right) \geq \vee_{\lambda} f^{-1}(\zeta_\lambda)$$

On the other hand, if $\zeta_1 = \vee_{\lambda} f^{-1}(\zeta_\lambda)$, then for all $\mu, \rho \in I^Y$ we have

$$\zeta_1\left(f^{-1}(\mu), f^{-1}(\rho)\right) \geq f^{-1}(\zeta_\lambda)(f^{-1}(\mu)\left(f^{-1}(\rho)\right) \geq \zeta_\lambda(\mu, \rho).$$

This together with (2) implies that $\zeta_1 \geq f^{-1}\left(\vee_{\lambda} \zeta_\lambda\right)$

(7) Let ζ be topogenous. Then

$$f^{-1}(\zeta)(\mu_1 \vee \mu_2, \rho) = \zeta(f(\mu_1) \vee f(\mu_2), 1 - f(1-\rho)$$
$$= \zeta(f(\mu_1) 1 - f(1-\rho \wedge)) \wedge \zeta(\mu_2), 1 - f(1-\rho))$$
$$= f^{-1}(\zeta)(\mu_1, \rho) \wedge f^{-1}(\zeta)(\mu_2, \rho)$$

Analogously, we get that $f^{-1}(\zeta)(\mu, \rho_1 \wedge \rho_2) = f_{-1}(\zeta)(\mu, \rho_1) \wedge f^{-1}(\zeta)(\mu, \rho_2)$.

(8) The proof is analogous to that of (7).

(5) By (3) and (7) we get that

$$\left[f^{-1}(\zeta)\right]^q \leq f^{-1}(\zeta^q)$$

On the other hand, let $\mu, \rho \in I^X$ and let $(\sigma_i), (\varphi_j)$ be finite families of fuzzy sets in Y with $f(\mu) = \vee \sigma_i, 1 - f(1-\rho) = \wedge \varphi_j$. Then $\mu \leq f^{-1}(f(\varpi)) \leq \vee f^{-1}(\sigma_i)$ and $\rho \geq f^{-1}(\varphi_j)$, and hence $f^{-1}(\zeta)^q(\mu, \rho) \geq f^{-1}(\zeta)^q\left(\vee_i f^{-1}(\sigma_i), \wedge_j f^{-1}(\varphi_j)\right)$

$$\geq \inf_{i,j} f^{-1}(\zeta)\left(f^{-1}(\sigma_i), f^{-1}(\varphi_j)\right)$$
$$\geq \inf_{i,j} \zeta(\sigma_i, \varphi_j)$$

If follows from this that

$$f^{-1}(\zeta)^q(\mu, \rho) \geq f^{-1}(\zeta^q)(\mu, \rho)$$

(5) The proof is analogous to that of (5).

(9) & (10) follow directly from the definitions.

(11) Let μ, ρ be fuzzy sets in X, $A \subset Y$ and $B = f^{-1}(A)$. Then

$$\zeta_2(\mu, B) = \zeta'\left(f(\mu), \left[f(B^C)\right]^C\right) \geq \zeta'(f(\mu), A)$$
$$\zeta_2(B, \rho) = \zeta(f(B), 1 - f(1-\rho) \geq \zeta(A, 1 - f(1-\rho))$$

Hence

$$\zeta o \zeta'(f(\mu), 1 - f(1-\rho)) = \sup_{A \subset Y} \zeta'(f(\mu), A) \wedge \zeta(A, 1 - f(1-\rho)$$
$$\leq \sup_{B \subset X} \zeta_2(\mu, B) \wedge \zeta_1(B, \rho)$$
$$= \zeta_1 o \zeta_2(\mu, \rho)$$

and so $\zeta_3 \leq \zeta_1 o \zeta_2$

Finally, let f be onto. For $A \subset X$ and $B = f(A)$, we have $\left[f(A^C)\right]^C \subset B$ and so

$$\zeta_2(\mu, A) \wedge \zeta_1(A, \rho)$$

$=\zeta'(f(\mu)),[f(A^C)]^C \wedge \zeta(f(A),1-f(1-\rho)$

$\leq \zeta'(f(\mu),B) \wedge \zeta(B,1-f(1-\rho))$

$=f^{-1}(\zeta o \zeta')(\mu,\rho)$

Thus, for f onto, we have $\zeta_1 o \zeta_2 \leq \zeta_3$ and the result follows.

Proposition. 3.2 :

Let $f:X \to Y$ and $g:Y \to Z$ be functions and let ζ be a fuzzy semi-topogenous order on Z.

Then $(gof)^{-1}(\zeta)=f^{-1}(g^{-1}(\zeta))$

Proof :

It follows directly from the previous definitions.

4. *Invere Image of a Fuzzy Syntopogenous Structure*

Proposition 4.1 :

Let $f:X \to Y$ be a function and let S be a fuzzy syntopogenous structure on Y. Then :

(1) $f^{-1}(S)=\{f^{-1}(\zeta):\zeta \in S\}$ is a fuzzy syntopogenous structure on X.

(2) If S is perfect, biperfect or symmetrical, then $f^{-1}(S)$ is perfect, biperfect or symmetrical, respectively.

(3) $f^{-1}(\tau(S))=\tau(f^{-1}(S))$

(4) If $(S_\lambda)_{\lambda \in \wedge}$ is a family of fuzzy syntopogenous structure on Y, then

$f^{-1}\left(\underset{\lambda}{\vee} S_\lambda\right)=\underset{\lambda}{\vee} f^{-1}(S_\lambda)$

Proof :

(1) and (2) follows directly from the definitions and from proposition (1).

(3) Let $\zeta=\zeta_s, \zeta_1=\zeta_{f^{-1}(S)}, \tau=\tau(S), \tau_1=\tau(f^{-1}(S))$. For a fuzzy subset of Y we have

$\mu^0(y)=\zeta_S(y,\mu)$.

Let now $\mu \in \tau$ and let ρ be the τ_1- interior of $f^{-1}(\mu)$.

Then

$\rho(x)=\zeta_1(x,f^{-1}(\mu))=\underset{\zeta \in S}{\mathrm{Sup}} f^{-1}((\zeta)(x,f^{-1}(\mu)))$

$$=\underset{\zeta\in S}{\mathrm{Sup}}\,\zeta\left(f(x),1-f(1-f^{-1}(\mu))\right)$$

$$\geq\underset{\zeta\in S}{\mathrm{Sup}}\,\zeta(f(x),\mu)$$

$$=\mu^{0}\left(f(x)\right)=\mu\left(f(x)\right)=f^{-1}(\mu)(x)$$

and so $\rho\geq f^{-1}(\mu)$, which implies that $f^{-1}(\mu)\in\tau_1$. Conversely, let $\sigma\in\tau_1$ and $\mu=1-f(1-\sigma)$.

If μ^0 is the $\tau-$ interior of μ, then

$$\sigma(x)=\zeta_1(x,\sigma)=\zeta_S\left(f((x),1-f(1-\sigma)\right)$$

$$=\zeta_S\left(f(x),\mu\right)=\mu^0\left(f(x)\right)=f^{-1}\left(\mu^0\right)(x).$$

Thus $\sigma=f^{-1}\left(\mu^0\right)\in f^{-1}(\tau)$.

Hence $\tau_1=f^{-1}(\tau)$

(4) Let $\lambda_1,\pi_2,\ldots\ldots.\lambda_n\in\wedge,\zeta_k\in S_{\lambda_k}$. Then

$$f^{-1}\left[(\zeta_1\vee\ldots\ldots\vee\zeta_n)^q\right]=\left[f^{-1}(\zeta_1)\vee\ldots..\vee f^{-1}(\zeta_n)\right]^q.$$

It follows from this that

$$f^{-1}\left(\vee S_\lambda\right)=\vee f^{-1}\left(S_\lambda\right)$$

5. *Continuity:-*

Definition. 5.1:

Let (S, S') be fuzzy topogenous structures on X, Y respectively, and let f be a function from X to Y. Then f is said to be (S, S') - continuous if $f^{-1}(S')$ is coarser than S.

Proposition 5.2 :

Let (X, S_1), (Y, S_2) and (Z, S_3) be fuzzy syntopogenous spaces and let $f:X\to Y$, $g:Y\to Z$ be functions. Then :

(1) If f is $S_1, S_2)$ – continuous then f is $(\tau(S_1),\tau(S_2))$ - continuous.

(2) If f is (S_1, S_2) – continuous and g is (S_2, S_3) – continuous, then go f is (S_1, S_3) – continuous

Proof :

(1) Since $f^{-1}(S_2)$ is coarser than S_1, we have $\tau\left(f^{-1}(S_2)\right)\subset\tau(S_1)$.

Since $\tau\left(f^{-1}(S_2)\right)=f^{-1}\left(\tau(S_2)\right)$, the result follows.

(2) It follows from the equality $(gof)^{-1}(S_3)=f^{-1}(g^{-1}(S_3))$.

We have also the following propositions without proof.

Proposition. 5.3 :

Let $\{(Y_\lambda,S_\lambda)\ \lambda\in\wedge\}$ be a family of fuzzy syntopogenous spaces, X a set and, for each $\lambda\in\wedge, f_\lambda : X\to Y_\lambda$ a function. If $S=\vee_{\lambda\in\wedge} f^{-1}(S_\lambda)$, then each f_λ is (S,S_λ)- continuous. Moreover, S is coarser than any fuzzy syntopogenous structure S' on X for which each f_λ is (S',S_λ) - continuous.

Proposition. 5.4 :

Let $(Y_{\lambda,\lambda})_{\lambda\in\wedge}, f_\lambda$ and S be as in the preceding proposition and let (Y, S') be a fuzzy syntopogenous space. Then a function $g: Y\to X$ is (S, S') - continuous iff each f_λ og is (S',S_λ)-continuous.

Proof :

The necessity follows from propositions (4) and (5). Conversely, let us assume that each f_λ og is (S',S_λ) - continuous and let $S'_\lambda = f_\lambda^{-1}(S_\lambda)$.

Then $g^{-1}(S'_\lambda) = \vee_{\lambda\in\wedge} (f_\lambda\, og)^{-1}(S_\lambda)$ is coarser than S'.

Since $g^{-1}(S) = \vee_{\lambda\in\wedge} g^{-1}(S'_\lambda)$,

We get that $g^{-1}(S)$ is coarser than S' and so g is (S', S) – continuous

Definition. 5.6 :

Let Y be a subset of X and let ζ be a fuzzy semi-topogenous order on X. The restriction ζ / Y of ζ to Y is defined to be the fuzzy semi-topogenous order $f^{-1}(\zeta)$ on Y, where $f : Y\to X$ is the inclusion map. If S is a fuzzy syntopogenous structure on X, then $S/Y = \{\zeta / Y : \zeta\in S\}$ is defined as the fuzzy syntopogenous induced by S on Y. If $f : Y \to X$ is the inclusion map, then $\tau(S/Y)=f^{-1}(\tau(S))=\tau(S)/Y$.

Proposition. 5.7 :

Let (X,S), (Y. S') be fuzzy syntopogenous spaces and let f be a function from X to Y. Then f is (S, S') – continuous iff $f : X \to f(X)$ is (S, S' / f (X))- continuous.

Proof :

The proof follow from proposition (6)

6. *Product Fuzzy Syntopogenous Spaces :*

Definition. 6.1 :

Let $(X_{\lambda,} S_{\lambda})_{\lambda\in\wedge}$ be a family of fuzzy syntopogenous spaces and let $X=\Pi_{\lambda\in\wedge}X_{\lambda}$. If π_{λ} denotes the canonical projection of X onto X_{λ}, then the fuzzy syntopogenous structure $\vee_{\lambda\in\wedge}\ \pi_{\lambda}^{-1}(S_{\lambda})=S$ is called the product of the family $(S_{\lambda})_{\lambda\in\wedge}$ and it will be denoted by $\Pi_{\lambda\in\wedge}S_{\lambda}$. The set X equipped with the product fuzzy syntopogenous structure is called the product of the family $(X_{\lambda}, S_{\lambda})_{\lambda\in\wedge}$.

We have now the following theorem with the help of theorem (6).

Proposition. 6.2 :

Let $(X_{\lambda}, S_{\lambda})_{\lambda\in\wedge}$, X and $S=\Pi_{\lambda\in\wedge}S_{\lambda}$ be as in definition (3). Then : (1) the fuzzy topology $\tau(S)$ coincides with the product of the fuzzy topologies $\tau(S_{\lambda}), \lambda\in\wedge$

(2) If g is a function from a fuzzy syntopogenous space (Y, S') to X, then g is (S', S) – continuous iff each π_{λ} og is (S', S_{λ}) continuous.

Proposition. 6.3 :

Let $(X_{\lambda}, S_{\lambda})_{\lambda\in\wedge}$ be a family of biperfect fuzzy syntopogenous space, $X=\Pi X_{\lambda}$ and $S=\Pi S_{\lambda}$. Then S^b is coarser than any biperfect fuzzy syntopogenous structure S' on X for which each π_{λ} is (S', S_{λ})-continuous.

Moreover $\tau(S^b)=\tau(S)=\Pi\tau(S_{\lambda})$.

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