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On Strong $(E, q)(C, 1)$ Summability of Fourier Series and its Derived Series

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Abstract

The present paper deals with theorem on strong $(E, q)(C,1)$ summability of Fourier series and its derived series under very general condition. It generalizes a well-known result on strong $(E,1)(C,1)$ summability of Fourier series under certain condition due to Bhatia and Sachan[9].

Keywords : $(E, q)(C,1)$ summability, Fourier series, Derived series, Convergence.

2020 Mathematics Subject Classifications: 40G15,40B05,40B08, 20F14.

1. Introduction

An infinite series $\sum u_n$ with the sequence $\{S_n\}$ of its partial sums is said to be summable $(C,1)$ to a fixed and finite sum S if the sequence-to-sequence transformation, $C_n^1 = \frac{1}{n+1}\sum_{m=0}^{n} S_m$ tends to S as $n \to \infty$ (Titchmarsh[3]) and the nth (E, q) mean, $q > 0$, of the sequence $\{S_n\}$ of partial sums of the series $\sum u_n$ is given by $E_n^q = \frac{1}{(q+1)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} S_k$. If $E_n^q \to S$ as $n \to \infty$, then the series $\sum u_n$ or the sequence $\{S_n\}$ of its partial sums is said to be summable (E, q) to the sum S. (Hardy [4],p.180).

Now superimposing (E, q) summability on the $(C,1)$ mean of the series $\sum u_n$ we get (E, q) $(C,1)$ summability of the series $\sum u_n$, where the n[th] (E, q) $(C,1)$ mean $(E_q C)_n^1$ of the sequence $\{S_n\}$ of partial sums of the series $\sum u_n$ is defined by the sequence-to-sequence transformation

$$(E_q C)_n^1 = \frac{1}{(q+1)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} C_k^1$$

Following the definition of strong $(C,1)$ summability given by Hardy and Littlewood[5], we define that the series $\sum u_n$ or the sequence $\{S_n\}$ of its partial sums is strongly summable $(E, q)\,(C,1)$ to the sum S if $\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{n-k}\left|C_k^1 - S\right| = o\left[(q+1)^n\right]$ as $n \to \infty$ (1.1)

The Fourier series of a 2π -periodic and Lebesgue integrable function of $f(t)$ in the interval $(-\pi, \pi)$ is given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \tag{1.2}$$

The series $$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t) \tag{1.3}$$

Which is obtained by differentiating term by term the series (1.2) with respect to t is known as the derived series of the Fourier series (1.2) of the function $f(t)$. Here

$$\sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) \tag{1.4}$$

is called as the conjugate series of the Fourier series (1.2) of the function $f(t)$.
Let us write at a point $t = x$, $\phi(t) = f(x+t) + f(x-t) - 2f(x)$, $g(t) = f(x+t) - f(x-t) - 2tf'(x)$, $\Phi(t) = \int_0^t |\phi(u)|du$, $G(t) = \int_0^t |dg(u)|$, where $f'(x)$ is the first derivative of $f(t)$ at $t = x$.

2. Preliminaries

The following lemmas are needed in order to prove our theorems,

Lemma 1. For $0 \le t \le \frac{1}{n}$

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right| = O\left[n(q+1)^n\right]$$

Proof. For $0 \le t \le \frac{1}{n}$,

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right|$$

$$\le \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} O(k)$$

$$= O(n) \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k}$$

$$= O[n(q+1)^n]$$

Lemma 2. For $t > \frac{1}{n}$

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right| = O\left[\frac{(q+1)^n}{nt^2}\right]$$

Proof. For $t > \frac{1}{n}$

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right|$$

$$= O(\frac{1}{nt^2}) \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k}$$

$$= O\left[\frac{(q+1)^n}{nt^2}\right]$$

Since Hardy and Littlewood[5] established Strong $(C,1)$ summability of Fourier series (1.2) under the condition

$$\Phi(t) = \int_0^t |\phi(u)| du = o(t) \text{ as } t \to 0 \tag{2.1}$$

Superimposing $(E,1)$ summability on the $(C,1)$ summability. Bhatia and Sachan[9] have recently studied strong $(E,1)$ $(C,1)$ summability of Fourier series (1.2) under the condition (2.1) by proving the following theorem :

Theorem A (Bhatia and Sachan[9]). If $f(x)$ is integrable (L) and $\Phi(t) = \int_0^t |\phi(u)| du = o(t)$, as $t \to 0$, then the Fourier series (1.2) of the function $f(t)$ is strongly summable $(E,1)$ $(C,1)$ to the sum $f(x)$ at $t = x$ in $(-\pi, \pi)$.

3. Main Results

In this section we prove very interesting result by superimposing (E,q), $q > 0$ on $(C,1)$ summability method and obtained a generalization of theorem A for strong (E,q) $(C,1)$ summability of the Fourier series (1.2) and also that of its derived series (1.3) by establishing following two theorems under very general condition.

Theorem 3.1. Let $\alpha(t)$ & $\beta(t)$ be any two positive function of t such that $\alpha(t)$, $\beta(t)$ and $t\dfrac{\alpha(t)}{\beta(t)}$ increase monotonically with t and let $\{p_n\}$ be a non-negative, monotonic, non-increasing sequence of constants with P_n such that its non-vanishing n^th^ partial sum $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$.

If $\alpha(n) = O[\beta(P_n)]$ as $n \to \infty$ and $\Phi(t) = \int_0^t |\phi(u)| du = o\left[\dfrac{\alpha\left(\frac{1}{t}\right)t}{\beta(P_\tau)}\right]$ as $t \to 0$, where τ is the integral part of $\dfrac{1}{t}$, then the Fourier series (1.2) of the function $f(t)$ is strongly summable $(E,q)(C,1)$ to the sum $f(x)$ at a point $t = x$ in $(-\pi, \pi)$.

Proof. The n^th^ $(C,1)$ mean $\sigma_n(x)$ of the sequence $\{S_n(x)\}$ of partial sums of the Fourier series (1.2) of a function $f(t)$ at the point $t = x$, is given by

$$\sigma_n(x) - f(x) = \frac{2}{n\pi} \int_0^{\delta} \frac{\phi(t)}{t^2} \sin^2 \frac{nt}{2} dt + o(1) \tag{3.1.1}$$

for $0 < \delta < \pi$ (Titchmarsh[3])

Following (1.1) for strong $(E,q)(C,1)$ summability of the Fourier series (1.2),we have

$$= \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left|\sigma_k(x) - f(x)\right|$$

$$= \frac{2}{\pi} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \int_0^{\delta} \frac{\phi(t)}{kt^2} \sin^2 \frac{kt}{2} dt \right| + o\left[\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \right]$$

$$\leq \frac{2}{\pi} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \int_0^{\delta} |\phi(t)| \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right| dt + o\left[(q+1)^n\right]$$

$$= O\left[\int_0^{\delta} |\phi(t)| \left\{ \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right| \right\} dt \right] + o\left[(q+1)^n\right]$$

$$= O\left[\int_0^{1/n} + \int_{1/n}^{\delta} \right] + o\left[(q+1)^n\right]$$

$$= O(I_1) + O(I_2) + +o\left[(q+1)^n\right] \text{, say} \tag{3.1.2}$$

Let us first consider

$$I_1 = \int_0^{1/n} |\phi(t)| \left\{ \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right| \right\} dt$$

$$= O\left[n(q+1)^n\right] \int_0^{1/n} |\phi(t)| dt \qquad \text{(using lemma 1)}$$

$$= O\left[n(q+1)^n\right] o\left[\frac{\alpha(n)}{n\beta(P_n)} \right] \qquad \text{(condition of theorem)}$$

$$= o\left[n(q+1)^n\right] \left[\frac{\beta(P_n)}{n\beta(P_n)} \right] \text{ (using condition on } \{p_n\} \text{)}$$

$$= o\left[(q+1)^n\right] \text{ as } \quad n \to \infty \tag{3.1.3}$$

Further considering I_2, we have

$$I_2 = \int_{1/n}^{\delta} |\phi(t)| \left\{ \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right| \right\} dt$$

$$= O\left[\frac{(q+1)^n}{n}\right] \int_{1/n}^{\delta} \frac{|\phi(t)|}{t^2} dt \qquad \text{(using lemma 2)}$$

$$= O\left[\frac{(q+1)^n}{n}\right] \left[\left(\frac{\Phi(t)}{t^2} \right)_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{\Phi(t)}{t^3} dt \right]$$

$$= O\left[\frac{(q+1)^n}{n}\right] \left[\left\{ \frac{\Phi(\delta)}{\delta^2} - n^2 \Phi\left(\frac{1}{n}\right) \right\} + \Phi\left(\frac{1}{n}\right) \int_{1/n}^{\delta} \frac{dt}{t^3} dt \right]$$

$$= o\left[(q+1)^n\right] - o\left[(q+1)^n n \Phi\left(\frac{1}{n}\right)\right] + o\left[\frac{\alpha(n)}{n\beta(P_n)}\right] O\left(n^2\right) \left[\frac{(q+1)^n}{n}\right]$$

$$= o\left[(q+1)^n\right] - o\left[(q+1)^n n \frac{\alpha(n)}{n\beta(P_n)}\right] + o\left[\frac{\alpha(n)}{\beta(P_n)} (q+1)^n\right]$$

$$= o\left[(q+1)^n\right] \qquad \text{as } n \to \infty \tag{3.1.4}$$

Collecting from (3.1.2) to (3.1.4) we get

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \sigma_k(x) - f(x) \right| = o\left[(q+1)^n\right] \qquad \text{as } n \to \infty$$

This completes the proof of Theorem 3.1.

Theorem 3.2. Let $\{p_n\}$ be as taken in theorem 3.1, together with $\alpha(t)$, $\beta(t)$ also the same, satisfying $\alpha(n) = \mathrm{O}[\beta(P_n)]$ as $n \to \infty$ and If $G(t) = \int_0^t |dg(u)| = o\left[\frac{\alpha\left(\frac{1}{t}\right)t}{\beta(P_\tau)}\right]$ as $t \to 0$, then the derived series (1.3) of the Fourier series (1.2) of the function $f(t)$ is strongly summable $(E,q)(C,1)$

to the sum $f'(x)$ at a point $t = x$ in $(-\pi, \pi)$.

Proof. Let $T_n(x)$, the sum of first n terms of the derived series (1.3) at a point $t = x$, we get

$$T_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{d}{dx}\left\{\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}\right]f(u)du$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} f(u)\left[\frac{d}{du}\left\{\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}\right]du$$

$$= -\frac{1}{2\pi}\int_0^{\pi}\left[f(x+t)-f(x-t)\right]\left[\frac{d}{dt}\left\{\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}\right]dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\, d\left[f(x+t)-f(x-t)\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\, dg(t) + f'(x)$$

Therefore

$$T_n(x) - f'(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\, dg(t) \tag{3.2.1}$$

The n^{th} $(C,1)$ mean of the sequence $\{T_n(x)\}$ of partial sums of the derived series (1.3) will be given by

Now let $$C_n^1(x) - f'(x) = \frac{1}{2n\pi}\int_0^{\pi} \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}} dg(t)$$

Where $$\sum_{k=0}^{n-1} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin \frac{1}{2}t} = \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}}$$

For $0 < \delta < \pi$ we have

$$C_n^1(x) - f'(x) = \frac{2}{n\pi}\int_0^{\delta} \frac{\sin^2 \frac{nt}{2}}{t^2} dg(t) + o(1) \tag{3.2.2}$$

Following (1.1) for strong $(E,q)(C,1)$ summability of the derived Fourier series (1.3), we have

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| C_k^1(x) - f'(x) \right|$$

$$= \frac{2}{\pi}\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \int_0^{\delta} \frac{\sin^2 \frac{kt}{2}}{kt^2} dg(t) \right| + o\left[\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k}\right]$$

$$\le \frac{2}{\pi}\int_0^{\delta} \left\{ \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| \frac{\sin^2 \frac{kt}{2}}{kt^2} \right| \right\} |dg(t)| + o\left[(q+1)^n\right]$$

$$= \mathrm{O}\left[\int_0^{\delta} |N_n(t)||dg(t)|\right] + o\left[(q+1)^n\right]$$

$$= O(J_1) + O(J_2) + o\left[(q+1)^n\right] \tag{3.2.3}$$

Now applying the condition in theorem 3.1, the lemma 1 and the condition of theorem 3.2 we can easily obtain, as in I_1, that

$$J_1 = o\left[(q+1)^n\right] \tag{3.2.4}$$

Further considering J_2 and following the lines of proof for I_2, we shall get

$$J_2 = o\left[(q+1)^n\right] \tag{3.2.5}$$

Collecting from (3.2.3) to (3.2.5), we shall immediately get the required result, i.e

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left| C_k^1(x) - f'(x) \right| = o\left[(q+1)^n\right], \quad \text{as} \quad n \to \infty$$

This completes the proof of Theorem 3.2.

4. Conclusion

In this paper, we have discussed the strong summability method of Fourier series and its derived series. Various results pertaining to the (E, q),(C,1) and product summability of Fourier series, its derived series and its conjugate series have been studied. The above theorems are for improvisation of many existing results of Fourier series and its derived series which extends and generalizes them. These results are motivation for many researchers in future to find better outcomes in the field of summability theory.

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