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JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES

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website:- www.ultrascientist.org**Polynomials Associated with Generalized Fibonacci Sequences**¹MANJEET SINGH TEETH and ^{2*}GARVITA AGRAWAL¹IPS Academy, Indore, Madhya Pradesh, India, 452001.²School of Mathematics, D.A.V.V. Indore, Madhya Pradesh, 452001 (India)

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Email Id : gravyagrawal@gmail.com manjeet.singh.t72@gmail.com<http://dx.doi.org/10.22147/jusps-B/380402>**Acceptance date 08th April 2026****Online Publication date 30th April 2026****Abstract**

In this paper we extend the Fibonacci recurrence relation to define the sequence $\{C_n\}$ and derive some properties of this sequence. We also define the four comparison sequence $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, $\{S_n\}$. We also obtain some identities with the help of generating matrix.

Key words: Fibonacci number, Fibonacci sequence.**Mathematics Subject Classification :** 11B39**Introduction**Wadill M. E.¹³, has extended the Fibonacci recurrence relation to define the sequence by

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, n \geq 3 \quad (1.1)$$

Where, K_0, K_1, K_2 are given arbitrary algebraic integer.Jaiswal D. V.⁹, has extended the Fibonacci recurrence relation to define the sequence by

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4}, n \geq 4 \quad (1.2)$$

Where, Q_0, Q_1, Q_2 are given arbitrary algebraic integers.

Harne S ⁸, has extended the Fibonacci recurrence relation to define the sequence $\{D_n\}$ by

$$D_n = D_{n-1} + D_{n-2} + D_{n-3} + D_{n-4} + D_{n-5}, n \geq 5 \quad (1.3)$$

Where, D_0, D_1, D_2 are given arbitrary algebraic integers.

Teeth M ¹², has extended the Fibonacci recurrence relation to define the sequence $\{C_n\}$ by

$$C_n = C_{n-1} + C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6}, n \geq 6 \quad (1.4)$$

Where, C_0, C_1, C_2 are given arbitrary algebraic integers.

In this paper, we shall further extend the Fibonacci recurrence relation^{3,4,5,6,7,11} to define the sequence $\{G_n\}$ and drive some properties of this sequence,

We shall also consider the five comparison sequences

$\{H_n\}, \{I_n\}, \{J_n\}, \{K_n\}$ and $\{L_n\}$

2 The Generalized Sequence $\{G_n\}$:

We consider the sequence, $\{G_n\} = G_0, G_1, G_2, \dots, G_n, \dots$

Where $G_0, G_1, G_2, G_3, G_4, G_5$, and G_6 are arbitrary algebraic integers all of which are not zero and

$$G_n = G_{n-1} + G_{n-2} + G_{n-3} + G_{n-4} + G_{n-5} + G_{n-6} + G_{n-7}, n \geq 7 \quad (2.1)$$

We also consider the sequence,

$$\{H_n\} = H_0, H_1, H_2, \dots, H_n, \dots$$

Where,

$$\begin{aligned} H_0 &= G_4 - G_3 - G_2 - G_1 - G_0, H_1 = G_5 - G_4 - G_3 - G_2 - G_1 \\ H_2 &= G_6 - G_5 - G_4 - G_3 - G_2, H_3 = G_7 - G_6 - G_5 - G_4 - G_3 \\ H_4 &= G_8 - G_7 - G_6 - G_5 - G_4, H_5 = G_9 - G_8 - G_7 - G_6 - G_5 \end{aligned} \quad (2.2)$$

With

$$H_n = G_{n-1} + G_{n-2} + G_{n-3} + G_{n-4} + G_{n-5} + G_{n-6}, n \geq 6 \quad (2.3)$$

And

$$\{I_n\} = I_0, I_1, I_2, I_n, \dots$$

Where,

$$I_0 = G_5 - G_4 - G_3 - G_2 - G_1 - G_0$$

$$I_1 = G_6 - G_5 - G_4 - G_3 - G_2 - G_1$$

$$\begin{aligned}
 I_2 &= G_7 - G_6 - G_5 - G_4 - G_3 - G_2 \\
 I_3 &= G_8 - G_7 - G_6 - G_5 - G_4 - G_3 \\
 I_4 &= G_9 - G_8 - G_7 - G_6 - G_5 - G_4
 \end{aligned}
 \tag{2.4}$$

With

$$I_n = G_{n-1} + G_{n-2} + G_{n-3} + G_{n-4} + G_{n-5}, n \geq 5
 \tag{2.5}$$

Now,

$$\{J_n\} = J_0, J_1, J_2, \dots, J_n,$$

Where,

$$\begin{aligned}
 J_0 &= G_6 - G_5 - G_4 - G_3 - G_2 - G_1 - G_0 \\
 J_1 &= G_7 - G_6 - G_5 - G_4 - G_3 - G_2 - G_1 \\
 J_2 &= G_8 - G_7 - G_6 - G_5 - G_4 - G_3 - G_2 \\
 J_3 &= G_9 - G_8 - G_7 - G_6 - G_5 - G_4 - G_3
 \end{aligned}
 \tag{2.6}$$

With

$$J_n = G_{n-1} + G_{n-2} + G_{n-3} + G_{n-4}, n \geq 4
 \tag{2.7}$$

Also,

$$\{K_n\} = K_0, K_1, K_2, \dots, K_n,$$

Where,

$$\begin{aligned}
 K_0 &= G_7 - G_6 - G_5 - G_4 - G_3 - G_2 - G_1 - G_0 \\
 K_1 &= G_8 - G_7 - G_6 - G_5 - G_4 - G_3 - G_2 - G_1 \\
 K_2 &= G_9 - G_8 - G_7 - G_6 - G_5 - G_4 - G_3 - G_2
 \end{aligned}
 \tag{2.8}$$

With,

$$K_n = G_{n-1} + G_{n-2} + G_{n-3}, n \geq 3
 \tag{2.9}$$

Next,

$$\{L_n\} = L_0, L_1, L_2, \dots, L_n,$$

Where,

$$\begin{aligned}
 L_0 &= G_8 - G_7 - G_6 - G_5 - G_4 - G_3 - G_2 - G_1 - G_0 \\
 L_1 &= G_9 - G_8 - G_7 - G_6 - G_5 - G_4 - G_3 - G_2 - G_1 \\
 L_2 &= G_{10} - G_9 - G_8 - G_7 - G_6 - G_5 - G_4 - G_3 - G_2
 \end{aligned}
 \tag{2.10}$$

With

$$L_n = G_{n-1} + G_{n-2}, n \geq 2 \quad (2.11)$$

Using (2.1) and (2.3) we have for $n \geq 13$

We have,

$$\begin{aligned} H_n &= (G_{n-2} + G_{n-7} + G_{n-8}) + (G_{n-3} + G_{n-8} + G_{n-9}) \\ &\quad + (G_{n-4} + G_{n-9} + G_{n-10}) \\ &\quad + (G_{n-5} + G_{n-10} + G_{n-11}) \\ &\quad + (G_{n-6} + G_{n-11} + G_{n-12}) \\ &\quad + (G_{n-7} + G_{n-12} + G_{n-13}) \end{aligned}$$

$$\begin{aligned} H_n &= H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6} + (G_{n-8} + G_{n-9} \\ &\quad + G_{n-12} + G_{n-13}) \end{aligned}$$

$$H_n = H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6} + H_{n-7}$$

On using (2.3) and (2.2) we have,

$$\begin{aligned} H_{12} &= (G_{10} + G_9 + G_8 + G_7 + G_6 + G_5) + (G_9 + G_8 + G_7 + G_6 + G_5 + G_4) \\ &\quad + (G_8 + G_7 + G_6 + G_5 + G_4 + G_3) + (G_7 + G_6 + G_5 + G_4 + G_3 + G_2) \\ &\quad + (G_6 + G_5 + G_4 + G_3 + G_2 + G_1) + (G_5 + G_4 + G_3 + G_2 + G_1 + G_0) \\ &\quad + (G_9 - G_8 - G_7 - G_6 - G_5) \end{aligned}$$

$$H_{12} = H_{11} + H_{10} + H_9 + H_8 + H_7 + H_6 + H_5$$

$$H_{11} = H_{10} + H_9 + H_8 + H_7 + H_6 + H_5 + H_4$$

$$H_{10} = H_9 + H_8 + H_7 + H_6 + H_5 + H_4 + H_3$$

$$H_9 = H_8 + H_7 + H_6 + H_5 + H_4 + H_3 + H_2$$

$$H_8 = H_7 + H_6 + H_5 + H_4 + H_3 + H_2 + H_1$$

Hence, we have for $n \geq 7$

$$H_n = H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6} + H_{n-7} \tag{2.12}$$

In the similar way, we have for $n \geq 7$

$$\begin{aligned} I_n &= (G_{n-2} + G_{n-3} - - - - - + G_{n-7} + G_{n-8}) \\ &\quad + (G_{n-3} + G_{n-4} - - - - - + G_{n-8} + G_{n-9}) \\ &\quad + (G_{n-4} - - - - - + G_{n-9} + G_{n-10}) \\ &\quad + (G_{n-5} - - - - - + G_{n-10} + G_{n-11}) \\ &\quad + (G_{n-6} - - - - - + G_{n-11} + G_{n-12}) \\ I_n &= I_{n-1} + I_{n-2} + I_{n-3} + I_{n-4} + I_{n-5} + I_{n-6} + I_{n-7} \end{aligned} \tag{2.13}$$

Proceeding on the same way, it can easily show that for $n \geq 7$ we have,

$$\begin{aligned} J_n &= (G_{n-2} + G_{n-3} - - - - - + G_{n-7} + G_{n-8}) \\ &\quad + (G_{n-3} + G_{n-4} - - - - - + G_{n-8} + G_{n-9}) \\ &\quad + (G_{n-4} - - - - - + G_{n-9} + G_{n-10}) \\ &\quad + (G_{n-5} - - - - - + G_{n-10} + G_{n-11}) \\ J_n &= J_{n-1} + J_{n-2} + J_{n-3} + J_{n-4} + J_{n-5} + J_{n-6} + J_{n-7} \end{aligned} \tag{2.14}$$

Similarly, we have for $n \geq 7$

$$\begin{aligned} K_n &= (G_{n-2} + G_{n-3} - - - - - + G_{n-7} + G_{n-8}) \\ &\quad + (G_{n-3} + G_{n-4} - - - - - + G_{n-8} + G_{n-9}) \\ &\quad + (G_{n-4} - - - - - + G_{n-9} + G_{n-10}) \\ K_n &= K_{n-1} + K_{n-2} + K_{n-3} + K_{n-4} + K_{n-5} + K_{n-6} + K_{n-7} \end{aligned} \tag{2.15}$$

Lastly, on similar lines we have for $n \geq 7$

$$\begin{aligned} L_n &= (G_{n-2} + G_{n-3} - - - - - + G_{n-7} + G_{n-8}) \\ &\quad + (G_{n-3} + G_{n-4} - - - - - + G_{n-8} + G_{n-9}) \\ L_n &= L_{n-1} + L_{n-2} + L_{n-3} + L_{n-4} + L_{n-5} + L_{n-6} + L_{n-7} \end{aligned}$$

Thus, the five sequences $\{H_n\}, \{I_n\}, \{J_n\}, \{K_n\}$ and $\{L_n\}$ are special cases of sequence $\{G_n\}$ are obtained by taking different initial values^{1,2,10}. On taking

$$\begin{aligned} G_0 = G_1 = G_2 = G_3 = 0, G_4 = G_5 = 1, G_6 = 2 \\ G_0 = G_1 = G_2 = 0, G_3 = 1, G_4 = 0, G_5 = 1, G_6 = 2 \\ G_0 = G_1 = 0, G_2 = 1, G_3 = 1, G_4 = 0, G_5 = 1, G_6 = 2 \\ G_0 = 0, G_1 = 1, G_2 = G_3 = G_4 = 0, G_5 = 1, G_6 = 2 \end{aligned}$$

$$G_0 = 1, G_1 = G_2 = G_3 = G_4 = 0, G_5 = 1, G_6 = 2$$

$$G_0 = G_1 = G_2 = G_3 = G_4 = 0, G_5 = 1, G_6 = 2$$

We have the following sequences

$$0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32, 64, \dots M_n, \dots$$

$$0, 0, 0, 1, 0, 1, 2, 4, 8, 16, 32, 63, \dots N_n, \dots$$

$$0, 0, 1, 0, 0, 1, 2, 4, 8, 16, 31, 62, \dots O_n, \dots$$

$$0, 1, 0, 0, 0, 1, 2, 4, 8, 15, 30, 60, \dots P_n, \dots$$

$$1, 0, 0, 0, 0, 1, 2, 4, 7, 14, 28, 56, \dots Q_n, \dots$$

$$0, 0, 0, 0, 0, 1, 2, 3, 6, 12, 24, 48, \dots R_n, \dots$$

Here we find that

$$N_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} + M_{n-5} + M_{n-6}$$

$$O_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} + M_{n-5}$$

$$P_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}$$

$$Q_n = M_{n-1} + M_{n-2} + M_{n-3}$$

$$R_n = M_{n-1} + M_{n-2}$$

Therefore, we can say that $\{M_n\}$ is a G_n type sequence, whereas $\{N_n\}$ is a H_n type sequence, while $\{O_n\}$ is a I_n type sequence, similarly, $\{P_n\}$ is a J_n type sequence, while $\{Q_n\}$ is a K_n type sequence, and $\{R_n\}$ is a L_n type sequence.

3 Linear Sums and Some Properties :

We have derived some properties of the sequence $\{G_n\}$, $\{H_n\}$, $\{I_n\}$, $\{J_n\}$, $\{K_n\}$ and $\{L_n\}$ expressing each of the terms $G_7, G_8, G_9, \dots, G_{n+6}$ as the sum of its seven preceding terms as given in (2.1) adding both sides we obtained on simplification -

$$\sum_{i=0}^n G_i = \frac{1}{6} \{G_{n+6} - G_{n+4} - 2G_{n+3} - 3G_{n+2} - 4G_{n+1} + G_n - (G_6 - G_4 - 2G_3 - 3G_2 - 4G_1 - 5G_0)\}$$

On using

$$\sum_{i=0}^n G_{7i} = \sum_{i=0}^{7n-1} G_i + G_0, \quad \sum_{i=0}^n G_{7i+2} = \sum_{i=0}^{7n+1} G_i + H_0, \quad \sum_{i=0}^n G_{7i+3} = \sum_{i=0}^{7n+2} G_i + I_0$$

$$\begin{aligned} \sum_{i=0}^n G_{7i+4} &= \sum_{i=0}^{7n+3} G_i + J_0, \quad \sum_{i=0}^n G_{7i+5} = \sum_{i=0}^{7n+4} G_i + K_0, \quad \sum_{i=0}^n G_{7i+6} = \sum_{i=0}^{7n+5} G_i + L_0 \\ \sum_{i=0}^n G_{7i+7} &= \sum_{i=0}^{7n+6} G_i + (L_1 - G_0), \quad \sum_{i=0}^n G_{7i+8} = \sum_{i=0}^{7n+7} G_i + (K_1 - G_0) \\ \sum_{i=0}^n G_{7i+9} &= \sum_{i=0}^{7n+8} G_i + (J_1 - G_0), \quad \sum_{i=0}^n G_{7i+10} = \sum_{i=0}^{7n+9} G_i + (I_1 - G_0) \\ \sum_{i=0}^n G_{7i+11} &= \sum_{i=0}^{7n+10} G_i + (H_1 - G_0) \end{aligned}$$

4. Property of sequence $\{M_n\}$:

Theorem 4.1 For the sequence $\{M_n\}$ we have,

$$\begin{vmatrix} M_n & M_{n+1} & M_{n+2} & M_{n+3} & M_{n+4} & M_{n+5} & M_{n+6} \\ M_{n+1} & M_{n+2} & M_{n+3} & M_{n+4} & M_{n+5} & M_{n+6} & M_{n+7} \\ M_{n+2} & M_{n+3} & M_{n+4} & M_{n+5} & M_{n+6} & M_{n+7} & M_{n+8} \\ M_{n+3} & M_{n+4} & M_{n+5} & M_{n+6} & M_{n+7} & M_{n+8} & M_{n+9} \\ M_{n+4} & M_{n+5} & M_{n+6} & M_{n+7} & M_{n+8} & M_{n+9} & M_{n+10} \\ M_{n+5} & M_{n+6} & M_{n+7} & M_{n+8} & M_{n+9} & M_{n+10} & M_{n+11} \\ M_{n+6} & M_{n+7} & M_{n+8} & M_{n+9} & M_{n+10} & M_{n+11} & M_{n+12} \end{vmatrix} = (-1)^{n+1}$$

Proof : consider the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

With the value (+1).

Next we have,

$$\Delta^2 = \begin{vmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

By Mathematical Induction

$$\Delta^n = \begin{vmatrix} M_{n+1} & N_{n+1} & O_{n+1} & P_{n+1} & Q_{n+1} & R_{n+1} & M_n \\ M_n & N_n & O_n & P_n & Q_n & R_n & M_{n-1} \\ M_{n-1} & N_{n-1} & O_{n-1} & P_{n-1} & Q_{n-1} & R_{n-1} & M_{n-2} \\ M_{n-2} & N_{n-2} & O_{n-2} & P_{n-2} & Q_{n-2} & R_{n-2} & M_{n-3} \\ M_{n-3} & N_{n-3} & O_{n-3} & P_{n-3} & Q_{n-3} & R_{n-3} & M_{n-4} \\ M_{n-4} & N_{n-4} & O_{n-4} & P_{n-4} & Q_{n-4} & R_{n-4} & M_{n-5} \\ M_{n-5} & N_{n-5} & O_{n-5} & P_{n-5} & Q_{n-5} & R_{n-5} & M_{n-6} \end{vmatrix}$$

On writing $R_{n+1} = M_n + M_{n-1}$, the R.H.S can be written as the sum of two determinants, one of which is obviously zero. Therefore,

$$\Delta^n = \begin{vmatrix} M_{n+1} & N_{n+1} & O_{n+1} & P_{n+1} & Q_{n+1} & M_{n-1} & M_n \\ M_n & N_n & O_n & P_n & Q_n & M_{n-2} & M_{n-1} \\ M_{n-1} & N_{n-1} & O_{n-1} & P_{n-1} & Q_{n-1} & M_{n-3} & M_{n-2} \\ M_{n-2} & N_{n-2} & O_{n-2} & P_{n-2} & Q_{n-2} & M_{n-4} & M_{n-3} \\ M_{n-3} & N_{n-3} & O_{n-3} & P_{n-3} & Q_{n-3} & M_{n-5} & M_{n-4} \\ M_{n-4} & N_{n-4} & O_{n-4} & P_{n-4} & Q_{n-4} & M_{n-6} & M_{n-5} \\ M_{n-5} & N_{n-5} & O_{n-5} & P_{n-5} & Q_{n-5} & M_{n-7} & M_{n-6} \end{vmatrix}$$

Also we have $Q_{n+1} = M_n + M_{n-1} + M_{n-2}$, the R.H.S can be written as the sum of three determinants, two of them are zero. Therefore,

$$\Delta^n = \begin{vmatrix} M_{n+1} & N_{n+1} & O_{n+1} & P_{n+1} & M_{n-2} & M_{n-1} & M_n \\ M_n & N_n & O_n & P_n & M_{n-3} & M_{n-2} & M_{n-1} \\ M_{n-1} & N_{n-1} & O_{n-1} & P_{n-1} & M_{n-4} & M_{n-3} & M_{n-2} \\ M_{n-2} & N_{n-2} & O_{n-2} & P_{n-2} & M_{n-5} & M_{n-4} & M_{n-3} \\ M_{n-3} & N_{n-3} & O_{n-3} & P_{n-3} & M_{n-6} & M_{n-5} & M_{n-4} \\ M_{n-4} & N_{n-4} & O_{n-4} & P_{n-4} & M_{n-7} & M_{n-6} & M_{n-5} \\ M_{n-5} & N_{n-5} & O_{n-5} & P_{n-5} & M_{n-8} & M_{n-7} & M_{n-6} \end{vmatrix}$$

Next, we have $P_{n+1} = M_n + M_{n-1} + M_{n-2} + M_{n-3}$, so that R.H.S can be written as the sum

of four determinants, three of them are zero. Therefore,

$$\Delta^n = \begin{vmatrix} M_{n+1} & N_{n+1} & O_{n+1} & M_{n-3} & M_{n-2} & M_{n-1} & M_n \\ M_n & N_n & O_n & M_{n-4} & M_{n-3} & M_{n-2} & M_{n-1} \\ M_{n-1} & N_{n-1} & O_{n-1} & M_{n-5} & M_{n-4} & M_{n-3} & M_{n-2} \\ M_{n-2} & N_{n-2} & O_{n-2} & M_{n-6} & M_{n-5} & M_{n-4} & M_{n-3} \\ M_{n-3} & N_{n-3} & O_{n-3} & M_{n-7} & M_{n-6} & M_{n-5} & M_{n-4} \\ M_{n-4} & N_{n-4} & O_{n-4} & M_{n-8} & M_{n-7} & M_{n-6} & M_{n-5} \\ M_{n-5} & N_{n-5} & O_{n-5} & M_{n-9} & M_{n-8} & M_{n-7} & M_{n-6} \end{vmatrix}$$

Now writing $O_{n+1} = M_n + M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}$, so that R.H.S can be written as the sum of five determinants, four of them are zero. Therefore,

$$\Delta^n = \begin{vmatrix} M_{n+1} & N_{n+1} & M_{n-4} & M_{n-3} & M_{n-2} & M_{n-1} & M_n \\ M_n & N_n & M_{n-5} & M_{n-4} & M_{n-3} & M_{n-2} & M_{n-1} \\ M_{n-1} & N_{n-1} & M_{n-6} & M_{n-5} & M_{n-4} & M_{n-3} & M_{n-2} \\ M_{n-2} & N_{n-2} & M_{n-7} & M_{n-6} & M_{n-5} & M_{n-4} & M_{n-3} \\ M_{n-3} & N_{n-3} & M_{n-8} & M_{n-7} & M_{n-6} & M_{n-5} & M_{n-4} \\ M_{n-4} & N_{n-4} & M_{n-9} & M_{n-8} & M_{n-7} & M_{n-6} & M_{n-5} \\ M_{n-5} & N_{n-5} & M_{n-10} & M_{n-9} & M_{n-8} & M_{n-7} & M_{n-6} \end{vmatrix}$$

Similarly, $N_{n+1} = M_n + M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} + M_{n-5}$ so that R.H.S can be written as the sum of five determinants, four of them are zero. Therefore,

$$\Delta^n = \begin{vmatrix} M_{n+1} & M_{n-5} & M_{n-4} & M_{n-3} & M_{n-2} & M_{n-1} & M_n \\ M_n & M_{n-6} & M_{n-5} & M_{n-4} & M_{n-3} & M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-7} & M_{n-6} & M_{n-5} & M_{n-4} & M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-8} & M_{n-7} & M_{n-6} & M_{n-5} & M_{n-4} & M_{n-3} \\ M_{n-3} & M_{n-9} & M_{n-8} & M_{n-7} & M_{n-6} & M_{n-5} & M_{n-4} \\ M_{n-4} & M_{n-10} & M_{n-9} & M_{n-8} & M_{n-7} & M_{n-6} & M_{n-5} \\ M_{n-5} & M_{n-10} & M_{n-10} & M_{n-9} & M_{n-8} & M_{n-7} & M_{n-6} \end{vmatrix}$$

On applying following elementary column operation

$C_2 \leftrightarrow C_7, C_3 \leftrightarrow C_6, C_4 \leftrightarrow C_5$ we have

$$\Delta^n = \begin{vmatrix} M_{n+1} & M_n & M_{n-1} & M_{n-2} & M_{n-3} & M_{n-4} & M_{n-5} \\ M_n & M_{n-1} & M_{n-2} & M_{n-3} & M_{n-4} & M_{n-5} & M_{n-6} \\ M_{n-1} & M_{n-2} & M_{n-3} & M_{n-4} & M_{n-5} & M_{n-6} & M_{n-7} \\ M_{n-2} & M_{n-3} & M_{n-4} & M_{n-5} & M_{n-6} & M_{n-7} & M_{n-8} \\ M_{n-3} & M_{n-4} & M_{n-5} & M_{n-6} & M_{n-7} & M_{n-8} & M_{n-9} \\ M_{n-4} & M_{n-5} & M_{n-6} & M_{n-7} & M_{n-8} & M_{n-9} & M_{n-10} \\ M_{n-5} & M_{n-6} & M_{n-7} & M_{n-8} & M_{n-9} & M_{n-10} & M_{n-11} \end{vmatrix}$$

Put $n - 11 = l$

We have

$$= (-1)^{l+11} \begin{vmatrix} M_{l+12} & M_{l+11} & M_{l+10} & M_{l+9} & M_{l+8} & M_{l+7} & M_{l+6} \\ M_{l+11} & M_{l+10} & M_{l+9} & M_{l+8} & M_{l+7} & M_{l+6} & M_{l+5} \\ M_{l+10} & M_{l+9} & M_{l+8} & M_{l+7} & M_{l+6} & M_{l+5} & M_{l+4} \\ M_{l+9} & M_{l+8} & M_{l+7} & M_{l+6} & M_{l+5} & M_{l+4} & M_{l+3} \\ M_{l+8} & M_{l+7} & M_{l+6} & M_{l+5} & M_{l+4} & M_{l+3} & M_{l+2} \\ M_{l+7} & M_{l+6} & M_{l+5} & M_{l+4} & M_{l+3} & M_{l+2} & M_{l+1} \\ M_{l+6} & M_{l+5} & M_{l+4} & M_{l+3} & M_{l+2} & M_{l+1} & M_l \end{vmatrix}$$

Rearranging the determinant and replace l by n , we get the required result.

5 Generating Matrix for $\{G_n\}$

In this section we obtain some identities with the help of generating matrix.

We consider the matrix

$$[S] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Using Mathematical Induction, we can show that

$$[S]^n = \begin{bmatrix} M_{n+1} & N_{n+1} & O_{n+1} & P_{n+1} & Q_{n+1} & R_{n+1} & M_n \\ M_n & N_n & O_n & P_n & Q_n & R_n & M_{n-1} \\ M_{n-1} & N_{n-1} & O_{n-1} & P_{n-1} & Q_{n-1} & R_{n-1} & M_{n-2} \\ M_{n-2} & N_{n-2} & O_{n-2} & P_{n-2} & Q_{n-2} & R_{n-2} & M_{n-3} \\ M_{n-3} & N_{n-3} & O_{n-3} & P_{n-3} & Q_{n-3} & R_{n-3} & M_{n-4} \\ M_{n-4} & N_{n-4} & O_{n-4} & P_{n-4} & Q_{n-4} & R_{n-4} & M_{n-5} \\ M_{n-5} & N_{n-5} & O_{n-5} & P_{n-5} & Q_{n-5} & R_{n-5} & M_{n-6} \end{bmatrix} \quad n \geq 6$$

and

$$[G_n, G_{n-1}, G_{n-2}, G_{n-3}, G_{n-4}, G_{n-5}, G_{n-6}] = [S]^n [G_6, G_5, G_4, G_3, G_2, G_1, G_0]$$

On using above two equations we have

$$\begin{bmatrix} G_{n+l} \\ G_{n+l-1} \\ G_{n+l-2} \\ G_{n+l-3} \\ G_{n+l-4} \\ G_{n+l-5} \\ G_{n+l-6} \end{bmatrix} = \begin{bmatrix} M_{n+1} & N_{n+1} & O_{n+1} & P_{n+1} & Q_{n+1} & R_{n+1} & M_n \\ M_n & N_n & O_n & P_n & Q_n & R_n & M_{n-1} \\ M_{n-1} & N_{n-1} & O_{n-1} & P_{n-1} & Q_{n-1} & R_{n-1} & M_{n-2} \\ M_{n-2} & N_{n-2} & O_{n-2} & P_{n-2} & Q_{n-2} & R_{n-2} & M_{n-3} \\ M_{n-3} & N_{n-3} & O_{n-3} & P_{n-3} & Q_{n-3} & R_{n-3} & M_{n-4} \\ M_{n-4} & N_{n-4} & O_{n-4} & P_{n-4} & Q_{n-4} & R_{n-4} & M_{n-5} \\ M_{n-5} & N_{n-5} & O_{n-5} & P_{n-5} & Q_{n-5} & R_{n-5} & M_{n-6} \end{bmatrix} \begin{bmatrix} G_n \\ G_{n-1} \\ G_{n-2} \\ G_{n-3} \\ G_{n-4} \\ G_{n-5} \\ G_{n-6} \end{bmatrix}$$

We have

$$G_{n+l} = M_{l+1}G_n + N_{l+1}G_{n-1} + O_{l+1}G_{n-2} + P_{l+1}G_{n-3} + Q_{l+1}G_{n-4} + R_{l+1}G_{n-5} + M_lG_{n-6}$$

Let us consider the matrix [Z] which is transpose of the matrix [S] i.e.

$$[Z] = [S]' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It can be easily seen that the sequence $G_5, H_6, I_6, J_6, K_6, L_6, G_6, \dots, G_{n-1}, H_n, I_n, J_n, K_n, L_n, G_n$ is generated by matrix [Z].

$$(5.1)$$

i.e.

$$[G_n, H_n, I_n, J_n, K_n, L_n, G_{n-1}] = [Z]^{n-6} [G_6, H_5, I_5, J_5, K_5, L_5, G_5], \quad n \geq 6 \quad (5.2)$$

On using (5.1) and (5.2), we get

$$\begin{aligned} & [G_{n+l}, H_{n+l}, I_{n+l}, J_{n+l}, K_{n+l}, L_{n+l}, G_{n+l-1}] \\ & = [Z]^{n+l-6} [G_6, H_5, I_5, J_5, K_5, L_5, G_5], \quad n \geq 6 \\ & = [Z]^l [G_n, H_n, I_n, J_n, K_n, L_n, G_{n-1}] \end{aligned}$$

$$= \begin{bmatrix} M_{l+1} & M_l & M_{l-1} & M_{l-2} & M_{l-3} & M_{l-4} & M_{l-5} \\ N_{l+1} & N_l & N_{l-1} & N_{l-2} & N_{l-3} & N_{l-4} & N_{l-5} \\ O_{l+1} & O_l & O_{l-1} & O_{l-2} & O_{l-3} & O_{l-4} & O_{l-5} \\ P_{l+1} & P_l & P_{l-1} & P_{l-2} & P_{l-3} & P_{l-4} & P_{l-5} \\ Q_{l+1} & Q_l & Q_{l-1} & Q_{l-2} & Q_{l-3} & Q_{l-4} & Q_{l-5} \\ R_{l+1} & R_l & R_{l-1} & R_{l-2} & R_{l-3} & R_{l-4} & R_{l-5} \\ M_l & M_{l-1} & M_{l-2} & M_{l-3} & M_{l-4} & M_{l-5} & M_{l-6} \end{bmatrix} \begin{bmatrix} G_n \\ H_n \\ I_n \\ J_n \\ K_n \\ L_n \\ G_{n-1} \end{bmatrix}$$

Hence,

$$G_{n+l} = M_{l+1}G_n + M_lH_n + M_{l-1}I_n + M_{l-2}J_n + M_{l-3}K_n + M_{l-4}L_n + M_{l-5}G_{n-1}.$$

$$H_{n+l} = N_{l+1}G_n + N_lH_n + N_{l-1}I_n + N_{l-2}J_n + N_{l-3}K_n + N_{l-4}L_n + N_{l-5}G_{n-1}.$$

$$I_{n+l} = O_{l+1}G_n + O_lH_n + O_{l-1}I_n + O_{l-2}J_n + O_{l-3}K_n + O_{l-4}L_n + O_{l-5}G_{n-1}.$$

$$J_{n+l} = P_{l+1}G_n + P_lH_n + P_{l-1}I_n + P_{l-2}J_n + P_{l-3}K_n + P_{l-4}L_n + P_{l-5}G_{n-1}.$$

$$K_{n+l} = Q_{l+1}G_n + Q_lH_n + Q_{l-1}I_n + Q_{l-2}J_n + Q_{l-3}K_n + Q_{l-4}L_n + Q_{l-5}G_{n-1}.$$

$$L_{n+l} = R_{l+1}G_n + R_lH_n + R_{l-1}I_n + R_{l-2}J_n + R_{l-3}K_n + R_{l-4}L_n + R_{l-5}G_{n-1}.$$

6. Conclusion

In this paper we define the sequence $\{G_n\}$ and its five comparison sequence $\{H_n\}$, $\{I_n\}$, $\{J_n\}$, $\{K_n\}$, $\{L_n\}$. We also derive linear sum and generating matrix for $\{G_n\}$.

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