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Subspaces and Regularly Open T_0 Spaces

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Abstract

Within this paper, questions concerning subspaces of regularly open T_0 spaces are investigated and resolved.

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1. Introduction

Given property P in a topological space, there are many questions that logically and historically arise. Included among those questions are questions concerning subspaces. Within classical topological studies, the question concerning subspaces for a property P has been "Does a space have property P iff every subspace has property P? Within this paper, properties for which the above statement is true are called subspace properties. In all cases for subspace properties, the proofs of the converse statement for the subspace theorem cited above are all the same and have nothing to do with the property: "Since the space is a subspace of itself and every subspace has the property, then the space has the property." As a result, proper subspace inherited properties were introduced and investigated[1] giving the properties themselves a new, major role in the investigation of subspace questions.

Definition 1.1. Let (X,T) be a space and let P be a property of topological spaces. If (X,T) has property P when every proper subspace of (X,T) has property P, then P is said to be a proper subspace inherited property[1].

Since singleton set topological spaces satisfy many topological properties, within the recent paper[1] only spaces with three or more elements were considered. Each of the subspace properties T_0, R_0, T_1, R_1, T_2, weakly Urysohn, Urysohn, regular, and T_3 proved to be proper subspace inherited properties and new characterizations for each of the properties were given.

Theorem 1.1. A space has property P iff every proper subspace of (X,T) has property P, where P can be each of the properties cited above[1].

The results above raised the question

of whether or not topological properties could be further characterized using only certain types of sets within the space. With the important role of open and closed sets in the study of topology, a nature place to start such an investigation would be with open sets and closed sets. The investigation of this question led to the new characterizations of T_0 spaces given below[2].

Theorem 1.2. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty closed set C, (C,T_C) is T_0, and (c) for each nonempty proper closed set C, (C,T_C) is T_0 and for x and y in X, x not y, Cl({x}) is not X or Cl({y}) is not X [2].

Theorem 1.3. Let (X.T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty closed set C in X, for each nonempty closed set Y in C, (Y,T_Y) is T_0, (c) for each nonempty closed set C in X, for each nonempty proper closed set Y in C, (Y,T_Y) is T_0, and for x and y in X, x not y, Cl({x}) is not X or Cl({y}) is not X, and (d) for each nonempty proper closed set C in X, for each nonempty proper closed set Y in C, (Y,T_Y) is T_0, and for each nonempty closed set W in X, for each x and y in W, x not y, there exits a proper closed set D in W such that x is in D or y is in D [2].

Theorem 1.4. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty open set O in X, (O,T_O) is T_0, and (c) for each nonempty proper open set O in X, (O,T_O) is T_0 and for each x and y in X, x not y, there exists a proper open set containing x or a proper open set containing y[2].

Theorem 1.5. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty open set O in X, for each open set U in O, (U,T_U) is T_0, (c) for each nonempty open set O in X, for each nonempty open proper subset U in O, (U,T_U) is T_0 and for each x and y in X, x not y, there exists an open proper set W in X such that x is in W or y is in W, and (d) for each nonempty proper open set O in X, for each nonempty proper open set U in O, (U,T_U) is T_0 and for each open set W in X and x and y in W, x not y, there exists a proper open set V in W such that x is in V or y is in V[2].

The successes above raised questions concerning the use of regularly open subsets and regularly closed subsets in the continued investigation of subspaces. Regularly open sets were introduced in 1937 [5].

Definition 1.5. Let (X,T) be a space and let A be a subset of X. Then A is regularly open iff A= Int(Cl(A)).

Within the 1937 paper [5] it was shown that the regularly open sets of a space (X,T) is a base for a topology Ts on X coarser than T. The space (X,T) is semiregular iff the regularly open sets of (X,T) is a base for T [5].

The introduction of regularly open sets led to the introduction of regularly closed sets.

Definition 1.6. Let (X,T) be a space and let C be a subset of X. Then C is regularly closed iff one of the following equivalent conditions is satisfied: (1) $X \setminus C$ is regularly open and (2) $C = Cl(Int(C))$[6].

The focus on regularly open and regularly closed sets led to the introduction of regularly open separation axioms, which included the regularly open T_0 separation axiom [3].

Definition 1.7. A space (X,T) is regularly open T_0 iff for x and y in X, x not y, there exists a regularly open set containing only one of x and y [3].

Within this paper subspace questions concerning regularly open T_0 spaces are investigated and resolved.

2. Characterizations of Regularly Open T_0 Spaces Using Subspaces:

As in the initial investigations cited above[1] and[2], all spaces will have three or more elements. Also, as was the case for the investigation[2], care will be taken for proper subspaces with more than one element to ensure the resulting subspace topology is not the indiscrete topology.

Within the paper[3], it was proven that a space (X,T) is regularly open T_0 iff (X,Ts) is T_0. The combination of this result with the results above gives the following corollary.

Corollary 2.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is regularly open T_0, (b) each subspace of (X,Ts) is T_0, and (c) each proper subspace of (X,Ts) is T_0.

Within the paper[4], it was proven that use of the semiregularization process produces at most one new topology. Below this result is combined with the results above to prove regularly open T_0 is a semiregularization space property, *i. e.*, a property simultaneously shared by a space (X,T) and its semiregularization space (X,Ts).

Theorem 2.1. Let (X,T) be a space. Then (a) (X,T) is regularly open T_0 iff (b) (X,Ts) is regularly open T_0.

Proof: (a) implies (b): Since (X,T) is regularly open T_0, then (X,Ts) is T_0. Since Ts = (Ts)s, then (X,(Ts)s) is T_0 and, by Corollary 2.1, (X,Ts) is regularly open T_0.
(b) implies (a): Since (X,Ts) is regularly open T_0, then, by Corollary 2.1, (X,(Ts)s) is T_0 and since (Ts)s = Ts, then (X,Ts) is T_0 and (X,T) is regularly open T_0.

Below the characterizations of T_0 spaces given above using subspaces are used to further characterize regularly open T_0 spaces.

Theorem 2.2. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is regularly open T_0, (b) for each nonempty Ts-closed set C, $(C,(Ts)_C)$ is T_0, (c) for each nonempty proper Ts-closed set C, $(C,(Ts)_C)$ is T_0 and for x and y in X, x not y, $Cl_{Ts}(\{x\})$ or $Cl_{Ts}(\{y\})$ is not X, (d) for each nonempty proper regularly closed set C, $(C,(Ts)_C)$ is T_0, and for x and y in X, x not y, $Cl_{Ts}(\{x\})$ or $Cl_{Ts}(\{y\})$ is not X, and (e) for each regularly closed set C, $(C,(Ts)_C)$ is T_0.

Proof. Clearly, from the results above, (a) implies (b), (b) implies (c), and (c) implies (d).
(d) implies (e): Suppose (X,T) is not regularly open T_0. Let x and y be in X, x not y, such that every regularly open set containing one of x and y contains both x and y. Then $Cl_{Ts}(\{x\}) = Cl_{Ts}(\{y\})$ and $Cl_{Ts}(\{x\}) = Cl_{Ts}(\{y\})$ is not X. Let $O = X \setminus Cl_{Ts}(\{x\})$. Then O is a nonempty Ts-open set. Let z be in O. Let U be a regular open set containing z that is a subset of O. Then $V = X \setminus U$ is a proper regularly closed set containing x and y and $(V,(Ts)_V)$ is T_0. Let W be an element of $(Ts)_V$ containing only one of x and y, say only x of x and y is in W. Let Z be in Ts such that W is the intersection of V and Z. Let Y be regularly open such that x is in Y, which is contained in Z. Then Y is a regularly open set containing only one of x and y, which is a contradiction. Therefore (X,T) is regularly open T_0 and for every nonempty Ts-closed set C, which includes all nonempty regularly closed sets, $(C,(Ts)_C)$ is T_0.
(e) implies (a): Since X is regularly closed, then $(X,(Ts)_X) = (X,Ts)$ is T_0 which implies (X,T) is regularly open T_0.

An obvious question that arose in the consideration of equivalent statements in Theorem 2.2 was what would happen if Cl_{Ts} in the statement of (d) was replaced by Cl_T, which is resolved below.

The following result follows immediately from Theorem 2.2 and the fact that for each x in a space (X,T), $Cl_T(\{x\})$ is a subset of $Cl_{Ts}(\{x\})$.

Corollary 2.2. Let (X,T) be regularly open T_0. Then for each nonempty proper Ts-closed set C, $(C,(Ts)_C)$ is T_0 and for x and y in X, x not y, $Cl_T(\{x\})$ or $Cl_T(\{y\})$ is not X.

The following example shows the converse of Corollary 2.2 is not true.

Example 2.1. Let **N** denote the natural numbers, let **Q** denote the rational numbers, and let X be the union of **N** and **Q**. Let $\mathbf{B}_{\{1,2\}} = \{O \subseteq \mathbf{N}: \{1,2\} \subseteq O$, and $\mathbf{N} \setminus O$ is finite$\}$, let $\mathbf{B}_3 = \{O \subseteq \mathbf{N}: \mathbf{N} \setminus O$ is finite and neither 1 nor 2 is in $O\}$, and let $\mathbf{B}_Q = \{\{x\} \cup O: x \in X \setminus \mathbf{N}$ and $O \in \mathbf{B}_{\{1,2\}} \cup \mathbf{B}_3\}$. Then **B**, the union of $\mathbf{B}_{\{1,2\}}$, $\mathbf{B}_3$, and $\mathbf{B}_Q$, is a base for a topology T on X, every Ts-open set containing one of 1 and 2 contains both 1 and 2, $Cl_T(\{x\})$ is not X for all x in X, and there are no nonempty proper Ts-closed sets.

Theorem 2.3. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is regularly open T_0, (b) for each nonempty Ts-closed set C, for each nonempty $(Ts)_C$-closed set Y in C, $(Y,(Ts)_Y)$ is T_0, (c) for each nonempty Ts-closed set C in X, for each nonempty proper $(Ts)_C$-closed set Y in C, $(Y,(Ts)_Y)$ is T_0 and for x and y in X, x not y, $Cl_{Ts}(\{x\})$ or $Cl_{Ts}(\{y\})$ is not X, (d) for each nonempty proper Ts-closed set C in X, for each nonempty proper $(Ts)_C$-closed set Y in C, $(Y,(Ts)_Y)$ is T_0 and for each nonempty Ts-closed set W in X, for x and y in W, x not y, there exists a proper $(Ts)_W$-closed set D in W such that x is in D or y is in D, (e) for each nonempty proper regularly closed set C in X, for each nonempty proper $(Ts)_C$-closed set Y in C, $(Y,(Ts)_Y)$ is T_0 and for each nonempty Ts-closed set W in X, for x and y in W, x not y, there exists a proper $(Ts)_W$-closed set D in W such that x is in D or y is in D, and (f) for each nonempty regularly closed set C in X, for each nonempty proper $(Ts)_C$-closed set Y in C, $(Y,(Ts)_Y)$ is T_0 and for x and y in X, x not y, $Cl_{Ts}(\{x\})$ is not X or $Cl_{Ts}(\{y\})$ is not X.

Proof: From the results above, (a) implies (b), which implies (c), which implies (d), which implies (e).

(e) implies (f): Let x and y be in X, x not y. Let D be a proper Ts-closed set in X such that x is in D or y is in D. Consider the case that only one of x and y is in D, say only x is in D. Then $O = X\backslash D$ is Ts-open and there exists a regularly open set U such that y is in $U \subseteq O$. Thus consider the case that both x and y are in D. Let F be a proper $(Ts)_D$-closed set in D such that x is in F or y is in F. If F contains only one of x and y, then F is a Ts-closed set in X and, as above, there exists a regularly open set containing only one of x and y. Hence, consider the case that both x and y are in F. Then x and y are in D. Let z be in $X\backslash D$ and let V be regularly open such that z is in $V \subseteq X\backslash D$. Then $C = X\backslash V$ is a proper regularly closed set in X, $E = C \cap F$ is a proper $(Ts)_C$-closed set in C, and both x and y are in E. Since $(E,(Ts)_E)$ is T_0, there exists a $(Ts)_E$-open set W containing only one of x and y, say y is in W. Let Z be Ts-open such that $W = Z \cap E$. Let S be regularly open such that y is in $S \subseteq Z$. Thus, in all cases, there exists a regular open set containing only one of x and y. Hence (X,T) is regularly open T_0. Then, by the argument above, statement (c) is true, which implies (f).

(f) implies (a): Suppose (X,T) is not regularly open T_0. Let x and y be in X such that every regularly open set containing one of x and y contains both x and y. Then $Y = Cl_{Ts}(\{x\}) = Cl_{Ts}(\{y\})$, which is a proper Ts-closed subset of the regularly open set X and $(Y,(Ts)_Y)$ is T_0, which is a contradiction. Thus (X,T) is regularly open T_0.

Based on the history of the subspace questions given above one would want to know if regularly open T_0 is a T-subspace property and/or a T-proper subspace inherited property, or a Ts-subspace property and/or a Ts-proper subspace property. Below examples are given showing regularly open T_0 has none of these subspace properties.

The space (X,T) in Example 2.1 is not regularly open T_0 and has no nonempty proper Ts-closed subspaces. Thus, trivially, each nonempty proper Ts-closed set subspace is regularly open T_0. Hence regularly open T_0 is not a proper Ts-closed subspace inherited property.

Example 2.1. Let $\{y_n\}$ be a strickly increasing sequence of negative real numbers converging to 0 with $y_1 = -1$, let $\{x_n\}$ be a strickly decreasing sequence of real numbers converging to 0, and let $X = \{x: x = y_n \text{ or } x = x_n$ for some natural number n, or $x = 0\}$. Then $\{\{x\}: x \text{ is not } -1 \text{ or } 0\} \cup \{\{-1\} \cup [(-1/n,0) \cap X]: n \text{ is a natural number}\} \cup \{(-1/n,1/m) \cap X: n \text{ and } m \text{ are natural numbers}\}$

is a base for a topology T on X and (X,T) is T_1 and regularly open T_0. Let $Y = [-1,0] \cap X$. Then Y is T-closed and Ts-closed, every T_Y-regularly open set containing -1 contains 0 and every T_Y-regularly open set containing 0 contains -1, and (Y,T_Y) is not regularly open T_0.

Thus the attention is now turned to questions concerning open sets.

3. Characterizations Using Open, Ts-Open, and Regularly Open Set Subspaces.

Within the paper [4], it was proven that for a space (X,T), for each open set O, $(Ts)_O = (T_O)s$ and thus $(O, (Ts)_O) = (O, (T_O)s)$. Below this result is combined with Theorems 1.4 to further characterize regularly open T_0 spaces.

Theorem 3.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is regularly open T_0, (b) for each T-open set O,(O,T_O) is regularly open T_0, (c) for each Ts-open set O, (O, T_O) is regularly open T_0, (d) for each Ts-open set O, $(O,(Ts)_O)$ is regularly open T_0, (e) for each regularly open set O, (O,T_O) is regularly open T_0, (f) for each nonempty proper T-open set O, (O,T_O) is regularly open T_0 and for x and y in X, x not y, there exists a proper Ts-open set containing x or a proper open set containing y, (g) for each Ts-open set O, $(O,(Ts)_O)$ is regularly open T_0 and for x and y in X, x not y, there exists a proper Ts-open set containing x or a proper Ts-open set containing y, (h) for each nonempty proper regular open set O, (O,T_O) is regularly open T_0 and for x and y in X, x not y, there exists a proper Ts- open set containing x or a proper Ts-open set containing y.

Proof: (a) implies (b): Let O be open in X. Since (X,T) is regularly open T_0, then (X,Ts) is T_0 and $(O,(Ts)_O)$ is T_0. Since $(Ts)_O = (T_O)s$, then $(O,(T_O)s)$ is T_0 and, by results from above, (O,T_O) is regularly open T_0.
Clearly (b) implies (c) and, by Theorem 2.1, (c) implies (d) and (d) implies (e).
(e) implies (f): Let O be a nonempty proper T-open set in X. Let x and y be in X, x not y. Since X is regularly open, $(X,T_X) = (X,T)$ is regularly open T_0 and there exists a regularly open set U containing only one of x and y. Then U is a proper Ts-open set containing only one of x and y. Since (a) is true, then, by the argument above, (b) is true, which together with the argument above in this proof, implies (f).
(f) implies (g): Let O be nonempty Ts-open. Let x and y be in O, x not y.

Consider the case that O = X. Let U be a proper Ts-open set containing x or containing y. Suppose that U contains only one of x and y, say x is in U. Let V be a regularly open set containing x and a subset of U. Then V is a regularly open set containing only one of x and y. Thus consider the case that both x and y are in U. Then U is a nonempty proper T-open set and (U,T_U) is regularly open T_0. Let W be a T_U-regularly open set containing only one of x and y. Then U is T-open and $\{U \cap Int_T(Cl_T(Z)): Z \text{ is } T_O\text{-open}\}$ is a base for $(Ts)_U$ on U [4]. Since $(Ts)_U = (T_U)s$, let Y be T_U-open such that $(U \cap Int_T(Cl_T(Y))) \subseteq W$. Then $Int_T(Cl_T(Y))$ is a T-regularly open set[4] containing only one of x and y. Hence (X,T) is regularly open T_0. Thus statement (c) is true and (f) implies (g).

Clearly (g) implies (h).

(h) implies (a): Suppose that (X,T) is not regularly open T_0. Let x and y be in X, x not y, such that every regularly open set containing one of x and y contains both x and y. Let W be a proper Ts-open set containing x or containing y, say x is in W. Let O be regularly open such that x is in O, which is a subset of W. Then O is a proper regularly open set containing both x and y and (O,T_O) is regularly open T_0. Let Y be a T_O-regularly open set containing only one of x and y. Then, by the argument above in (f) implies (g), there exists a T-regularly open set containing only one of x and y, which is a contradiction. Thus (X,T) is regularly open T_0.

In a manner similar to the use of Theorem 2.2 and other results in this paper to extend Theorem 1.3 to Theorem 2.3, Theorem 3.1 can be used to extend to Theorem 1.5 to obtain additional characterizations of regularly open T_0 using open, Ts-open, and regularly open subspaces of subspaces and is left as a creative exercise for the interested reader.

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