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Fox's H-Function and Persistence of Equilibrium States in Biological Systems

S. S. SHRIVASTAVA

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Abstract

In an attempt to give extensions of the results in the theory of special functions we derive a proper integral whose integrand contains Fox's H-function having general argument and used it in finding out a solution of a non-linear differential equation

$$u_t = Du_{xx} + f(u)$$

related to a reaction-diffusion problems in biological models by means of linearization procedure.

1. Introduction

The H-function introduced by Fox[1], will be represented and defined in the following manner:

$$H_{p,q}^{m,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] = (1/2\pi\omega)\int_L \theta(s)\, x^s\, ds \tag{1.1}$$

where $\omega = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$$

For the nature of contour L in (1.1), the convergence, existence conditions and other details of the H-function, one can refer to [6].

New and broad aspects of diffusion problems in biological models, for example the persistence of equilibrium states in systems governed by the reaction-diffusion systems are investigated in this paper. We are to find whether those states, during the equilibrium state of the system and when the system is slightly perturbed or displaced, does the system returns to that state or does it evolve to a different[2] state?

For this purpose, in the interval $0 < x < L$ consider the reaction-diffusion equation[3]

$$u_t = Du_{xx} + f(u) \tag{1.2}$$

with no flux boundary conditions $u_x(0, t) = u_x(L, t) = 0$ (1.3)

and initial condition $u(x, 0) = \phi(x)$ (1.4)

Let $u(x, t) = u_e$ be a constant equilibrium solution, so that $f(u_e) = 0$. To fix the idea let $a = f'(u_e) > 0$. Obviously, the equilibrium solution satisfies the PDE and the boundary conditions. If U(x, t) be a small perturbation from the equilibrium solution, *i.e.*

$$u(x, t) = u_e + U(x, t), \qquad (1.5)$$

then the perturbation equation is determined by substituting this expression into (1.2), (1.3) and (1.4):

$$U_t = DU_{xx} + f(u_e + U) \qquad (1.6)$$
$$U_x(0, t) = U_x(L, t) = 0 \qquad (1.7)$$
$$U(x, 0) = \psi(x) \qquad (1.8)$$

2. Linearized Procedure:

In general, we cannot solve the perturbation equation (1.6) with conditions (1.7) and (1.8), because it is usually non-linear. At this stage, there is another argument we use called Linearization, to make conclusion about stability[4].

Since perturbation U is assumed small, so that the $U^2(x, t)$ term is very small compared to the U(x, t) term. Discarding the non-linear term, we obtained a linearized perturbation equation

$$U_t = DU_{xx} + aU, \quad a = f'(u_e) > 0 \qquad (2.1)$$

subject to the conditions

$$U_x(0, t) = U_x(L, t) = 0 \qquad (2.2)$$

and $$U(x, 0) = \psi(x) \qquad (2.3)$$

3. Proper Integral:

The following integral has been established in the paper:

$$\int_0^L (\sin \pi x/L)^{\omega-1} \cos(k\pi/L)\, H^{m,n}_{p,q}\left[z(\sin \pi x/L)^{2\rho}\Big|{(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right]dx$$

$$= (L/\sqrt{\pi})\, H^{m,n+2}_{p+2,q+2}\left[z\Big|{(1-\omega/2,\rho),(1/2-\omega/2,\rho)(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q},(1/2-\omega/2\pm k/2,\rho)}\right], \qquad (3.1)$$

provided that $\rho > 0$, $\mathrm{Re}[w + 2\rho \min_{1 \le j \le m} (b_j/\beta_j)] > 0$, $|\arg z| < \frac{1}{2} A\pi$; A, $\delta > 0$ (where A, δ are defined[6] in).

The integral (3.1) can be obtained easily by making use of the definition of H-function as given in section 1, the result [2]

$$\int_0^L (\sin \pi x/L)^{\omega-1} \cos(k\pi x/L)\, dx = \frac{L \cos(\frac{1}{2} k\pi)\, \Gamma(\omega)}{2^{\omega-1}\, \Gamma\{\frac{1}{2}(\omega \pm k+1)\}}, \qquad (3.2)$$

Re (ω) > 0 and Legendre's duplication formula

$$\sqrt{\pi}\, \Gamma(2z) = 2^{2Z-1}\, \Gamma(z)\, \Gamma(z + \tfrac{1}{2}).$$

4. Solution of the Problem:

The solution of the problem to be obtained is

$$U(x,t) = (2/\sqrt{\pi}) \sum_{k=0}^{\infty} [1-(1/2k\pi)\cos 2k\pi]^{-1} e^{\lambda_k t} \cos(k\pi/L) x$$

$$\times H^{m,n+2}_{p+2,q+2}\left[z\Big|{(1-\omega/2,\rho),(1/2-\omega/2,\rho)(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q},(1/2-\omega/2\pm k/2,\rho)}\right], \qquad (4.1)$$

where all conditions of convergence are same as in (3.1) and λ_k's are the roots of the equations $\lambda_k = a - Dk^2\pi^2/L^2$, k = 0, 1, 2, ...

Proof:

We may assume the general solution of the linearized perturbation equation (2.1) to be[3]:

$$U(x,t)=\sum_{k=0}^{\infty} C_k e^{\lambda_k t}\cos(k\pi/L)x, \quad (4.2)$$

where λ_k are the eigen values given by

$$\lambda_k = a - Dk^2\pi^2/L^2,\ k = 0, 1, 2, \ldots \quad (4.3)$$

Let

$$\psi(x)=(\sin \pi x/L)^{\omega-1} H^{m,n}_{p,q}\left[z(\sin \pi x/L)^{2\rho}\left|{(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right.\right]dx \quad (4.4)$$

If t = 0, then by virtue of (2.3), (4.2) and (4.4), we have

$$(\sin \pi x/L)^{\omega-1} H^{m,n}_{p,q}\left[z(\sin \pi x/L)^{2\rho}\left|{(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right.\right]dx,$$

$$=\sum_{k=0}^{\infty} C_k \cos(k\pi/L)x, \quad (4.5)$$

Multiply both sides of (4.5) by cos (hπ/L)x and integrate with respect to x from 0 to L. Now use the integral (3.1) and orthogonal property of cosines[5]

$$\int_0^L \cos(k\pi/L)x \cos(k\pi/L)x\, dx = \begin{cases} 0 & ; h \neq k \\ \tfrac{1}{2}[1-(1/2k\pi)\cos 2k\pi]; & h=k \end{cases} \quad (4.6)$$

we thus obtain the value of constant C_h. Substituting the value of C_h in (4.2), we get the desired result (4.1).

5. Special Cases:

1. Consider a large, canal of length L that contains a chemical of concentration c(x, t) dissolved in water of the concentration of the chemical satisfies the problem

$$C_t = D\, C_{xx},\ 0 < x < L$$

with no flux bounding conditions (1.3) and (1.4), where D is the diffusion constant, then the concentration C(x, t) in terms of Fox's H-function is given by

$$C(x,t)=(2/\sqrt{\pi})\sum_{k=0}^{\infty}[1-(1/2k\pi)]^{-1} e^{\lambda_k t}\cos(k\pi/L)x,$$

$$\times H^{m,n}_{p,q}\left[x\left|{(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right.\right] \quad (5.1)$$

where all conditions of convergence are specified with (3.1).

2. On taking m = q = 1, n = p = 0, $b_1 = 0$, $\beta_1=1$ in (4.1) reduce H-function in the exponential form [4, p. 151] *i.e.*

$$H^{1,0}_{0,1}[x|(0,1)] = e^{-z}$$

we thus get

$$U(x,t)=(2/\sqrt{\pi})\sum_{k=0}^{\infty}[1-(1/2k\pi)]^{-1} e^{\lambda_k t}\cos(k\pi/L)x,$$

$$\frac{\Gamma(\omega/2)\Gamma(\omega/2+1/2)}{\Gamma(\omega/2\pm k/2+1/2)}\ {}_2F_2\left[{\omega/2,\omega/2+1/2 \atop \omega/2+k/2\pm 1/2,};-z\right] \quad (5.2)$$

3. Further, putting z = 0 in (5.2), we obtain the following solution

$$U(x,t)=(2/\sqrt{\pi})\sum_{k=0}^{\infty}[1-(1/2k\pi)]^{-1} e^{\lambda_k t}\cos(k\pi/L)x, \quad (5.3)$$

6. Conclusion

Since the general solution (4.1) to the BVP (2.1) – (2.3) and its special cases (5.1) to (5.3) are linear combination of the modal solutions, hence it depends on all of the modes, *i.e.* it will decay if all of the modes decay, and it will grow if one of the modes grows. The eigen values λ_k, determine growth or decay.

Hence forth, we conclude that if the modes of low frequency k or low diffusion constant D or in systems of large size L, the system is instable, otherwise the system will be locally stable.

The reaction-diffusion biological models have been used to explain the evolution of form and structure in developmental biology, tumor growth, ecological spatial pattern, aggregation patterns on animal coats and many other processes in molecular and cellular biology.

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