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Unsteady flow of Walter's visco-elastic fluid through porous medium in a long uniform rectangular channel

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Abstract

The unsteady flow of Walter's visco-elastic fluid through porous medium in a long uniform straight channel of rectangular cross-section under the influence of time varying pressure gradient has been studied. The exact solution for the velocity of fluid has been obtained by using integral transform technique. Some particular cases of pressure gradient have been discussed in detail. Also we have discussed the case when porous medium is withdrawn. Besides, the corresponding viscous flow problem has been derived as a limiting case when the relaxation time parameter tends to become zero. We have also derived the case when porous medium is withdrawn i.e. if $K = \infty$.

Introduction

The problem of viscous fluid motion subjected to uniform and periodic body force for a finite time has been investigated by Dutta[2]. The flow of visco-elastic fluid between two parallel plates under the influence of uniform, exponential or periodic pressure gradient has been discussed by Pal and Sengupta[6], Roy, Sen and Lahiri[8], Das[1], Kundu and Sengupta[3], Kumar, Singh and Sharma[5] and others. Sengupta and Banerjee[9] studied the unsteady MHD flow of visco-elastic Rivlin-Ericksen and Walter's fluid through a straight tube. Kumar, Gupta and Jain[4] and Rajput, Mishra and Varshney[7] have considered the flow problems concerned with the walter's fluid.

In the present paper, the unsteady flow of Walter's visco-elastic fluid through porous medium in a long uniform rectangular channel under the influence of time dependent pressure gradient has been studied. Various particular cases have also been discussed in detail. We have also derived the case when porous medium is withdrawn *i.e.* if $K = \infty$.

Formulation of the Problem :

Here we are considering the motion

of visco-elastic Walter's fluid through porous medium inside a long uniform rectangular tube.

The boundary walls of the rectangular tube considered to be the planes $x = \pm a$, $y = \pm b$. The motion is under the influence of time dependent pressure gradient. Let the motion of the fluid along z-axis *i.e.* along the axis of rectangular channel.

Here, the corresponding Navier-Stockes equation motion for visco-elastic Walter's fluid through porous medium is given by

$$\frac{\partial W}{\partial t} = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial z} + \nu\left(1-\mu_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right) - \frac{\nu W}{K}, \tag{1}$$

where $W(x, y, t)$ is the velocity of the fluid in z-direction, μ_1 the kinematical coefficient of visco-elasticity, ρ the density of the fluid, $\nu\left(=\frac{\mu}{\rho}\right)$ the coefficient of viscosity and K is the permeability of porous medium.

Introducing the following non-dimensional quantities :

$$x^* = \frac{x}{a}, y^* = \frac{y}{a}, z^* = \frac{z}{a}, t^* = \frac{\nu}{a^2}t, p^* = \frac{a^2}{\rho\nu^2}p$$

$$W^* = \frac{a}{\nu}W, \mu_1^* = \frac{\nu}{a^2}\mu_1, K^* = \frac{1}{a^2}K$$

in equation (1), we get (after dropping stars)

$$\frac{\partial W}{\partial t} = -\frac{\partial p}{\partial z} + \left(1-\mu_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right) - \frac{1}{K}W, \tag{2}$$

Here, the initial and boundary conditions are

$$W(x, y, 0) = 0 \tag{3}$$

$$\left.\begin{aligned} &W(1, y, t) = 0, \quad 0 \le y \le l, \quad t > 0,\\ &\frac{\partial W}{\partial x} = 0; \qquad x = 0, \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} &W(x, l, t) = 0, \quad 0 \le x \le 1, \quad t > 0,\\ &\frac{\partial W}{\partial x} = 0; \qquad y = 0, \end{aligned}\right\} \tag{5}$$

where $l = \frac{b}{a}$.

Solution of the problem :

For solving eqn. (2), we use the following finite Fourier cosine transforms defined as :

$$W_c(i, y, t) = \int_0^1 W(x, y, t)\cos(p_i x)\, dx, \tag{6}$$

$$W_{\bar{c}}(x, j, t) = \int_0^l W_c(x, y, t)\cos(p_j y)\, dy, \tag{7}$$

where

$$p_i = (2i+1)\frac{\pi}{2}, \qquad p_j = (2j+1)\frac{\pi}{2l}.$$

Consequently, we have the following inverse of finite Fourier cosine transforms :

$$W(x, y, t) = 2\sum_{i=0}^{\infty} W_c(i, y, t)\cos(p_i x), \tag{8}$$

$$W_c = (i, y, t) = \frac{2}{l}\sum_{j=0}^{\infty} W_{\bar{c}}(i, j, t)\cos(p_j y), \tag{9}$$

We use transforms (6) and (7) to initial condition (3), we get

$$W_{\bar{c}}(i, j, 0) = 0 \tag{10}$$

Also taking finite Fourier cosine transform to boundary conditions, we have

$$\left.\begin{aligned} &W_c(i, l, t) = 0 \\ &\frac{\partial W_c}{\partial y}(i, 0, t) = 0 \end{aligned}\right\} \tag{11}$$

Applying transforms (6) and (7) to the equation of motion (2) and using initial and boundary conditions (10) and (11), we get

$$\zeta \frac{\partial W_{\bar{c}}}{dt} + \xi_1 W_{\bar{c}} = \frac{(-1)^{i+j} F(t)}{p_i p_j}, \tag{12}$$

where

$$W_{\bar{c}} = \int_0^1 \int_0^l W(x, y, t) \cos(p_i x) \cos(p_j y) dx\, dy,$$

$$\frac{\partial p}{\partial z} = -F(t),$$

$$\zeta = 1 - \mu_1(p_i^2 + p_j^2)$$

and $\xi_1 = \dfrac{1}{K} + p_i^2 + p_j^2.$

Then using the Laplace transform defined as :

$$\left.\begin{aligned} &\bar{W}_{\bar{c}}(s) = \int_0^\infty W_{\bar{c}} e^{-st} dt, \\ &\text{and} \\ &\bar{F}(s) = \int_0^\infty F(t) e^{-st} dt, \end{aligned}\right\} \tag{13}$$

and, by condition (11) on equation (12), we get

$$\zeta s \bar{W}_{\bar{c}} + \xi_1 \bar{W}_{\bar{c}} = \frac{(-1)^{i+j} \bar{F}(s)}{p_i p_j} \tag{14}$$

Now, by Laplace inversion formula and using convolution theorem, we get,

$$W_{\bar{c}} = \frac{(-1)^{i+j}}{p_i p_j \zeta} \int_0^t F(t-\lambda) e^{-(\xi_1/\zeta)\lambda} d\lambda. \tag{15}$$

Thus, by Fourier cosine inversion formula as in equation (8) and (9), the expression of velocity becomes

$$W(x, y, t) = \frac{4}{l} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{(-1)^{i+j}}{p_i p_j \eta} \left\{ \int_0^t F(t-\lambda) e^{-c_1 \lambda} d\lambda \right\} \times \cos(p_i x) \cos(p_j y) \right], \tag{16}$$

where

$$c_1 = \frac{\xi_1}{\zeta},\ p_i = (2i+1)\frac{\pi}{2},\ p_j = (2j+1)\frac{\pi}{2l},$$

We discuss the nature of velocity for following different particular cases :

Case I : Flow under constant pressure gradient: Let,

$$F(t) = F_0 \qquad \text{(a constant).}$$

From equation (16), the velocity will be

$$W = \frac{4}{l} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{(-1)^{i+j} F_0}{p_i p_j \xi_1} (1 - e^{-c_1 t}) \cos(p_i x) \cos(p_j y) \right]. \tag{17}$$

Case II : Flow under impulsive pressure gradient :
Let,

$$F(t) = f_0 \delta(t),$$

where $\delta(t)$ is the unit impulse function defined as

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

So, from equation (16), we get the velocity

$$W = \frac{4}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{(-1)^{i+j} f_0}{p_i p_j \zeta} e^{-c_1 t}\cos(p_i x)\cos(p_j y)\right]. \quad (18)$$

Case III : Flow under transient pressure gradient :

Let,

$$F(t) = f_1 e^{-Nt}, (N > 0),$$

where f_1 is a constant.

So, from equation (16), the velocity takes form

$$W = \frac{4}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{(-1)^{i+j} f_1 e^{-Nt}}{p_i p_j(\xi_1 - N\zeta)}\left\{1 - e^{-(c_1 - N)t}\right\} \times \cos(p_i x)\cos(p_j y)\right]. \quad (19)$$

Case IV : Flow under periodic pressure gradient :

Let,

$$F(t) = \mathrm{Re}(F_1 e^{i\omega t}),$$

where F_1 is a constant,

From equation (16), the velocity becomes

$$W = \frac{4}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{(-1)^{i+j} F_1}{p_i p_j(\xi_1^2 + \omega^2\zeta^2)}\left\{\omega\zeta\sin\omega t + \xi_1(\cos\omega t - 1)\right\}\right] \times \cos(p_i x)\cos(p_j y) \quad (20)$$

Case V : When the fluid is purely viscous :

For purely viscous fluid the kinematical co-efficient of visco-elasticity μ_1=0 and we get,

$$W = \frac{4}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{(-1)^{i+j}}{p_i p_j}\int_0^t F(t-\theta)e^{-\xi_1\theta}d\theta\right] \times \cos(p_i x)\cos(p_j y), \quad (21)$$

where

$$\xi_1 = \frac{1}{K} + p_i^2 + p_j^2, \qquad c_1 = \xi_1,$$

Case VI : When porous medium is withdrawn i.e. $K = \infty$.

We get all results for fluid motion in the absence of porous medium.

The values of ξ_1 and ζ are given by

$\xi_1 = p_i^2 + p_j^2$ and

$$\zeta = 1 - \mu_1(p_i^2 + p_j^2) = 1 - \mu_1\xi_1 \quad (22)$$

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