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Doubly interspersed sequences for Fibonacci numbers

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Abstract

Distinct positive Fibonacci sequences either eventually intersperse or else doubly intersperse. Necessary and sufficient condition for the latter case is given and each pair of distinct Fibonacci sequences eventually intersperse or doubly intersperse .We discuss that each 3-tuple of distinct Fibonacci sequences eventually intersperse or doubly intersperse.

1. Introduction

Interspersions among Fibonacci sequences can be arranged as array of numbers with some special conditions. A Stolarsky interspersion[4] is an array of positive integers, each occurring exactly once, such that the rows of the array are Fibonacci sequences, each pair of which eventually intersperse and such that the first column of the array is increasing. Among Stolarsky Interspersions are the original Stolarsky array[6], the Wythoff array[5], the golden array[3] and the Wythoff dual array[3]. C. Kimberling[1] gave an idea that each pair of distinct Fibonacci sequences eventually intersperse or doubly intersperse.

Sequences $u=(u_1,u_2,.........\,...)$, $x=(x_1,x_2,.........\,..)$, $y=(y_1,y_2,..........\,..)$ eventually intersperse if there exist i , j & k such that

$$0<u_i<x_j<y_k<u_{i+1}<x_{j+1}<y_{k+1}<u_{i+2}<x_{j+2}<y_{k+2}<.......... \quad (1.1)$$

i.e., if $u_{i+h}<x_{j+h}<y_{k+h}$ for all $h\geq 0$. Sequences u, x and y eventually doubly intersperse if there exists I, j & k such that

$$0<u_i<u_{i+1}<x_j<x_{j+1}<y_k<y_{k+1}<u_{i+2}<u_{i+3}<x_{j+2}<x_{j+3}<y_{k+2}<y_{k+3}<...... \quad (1.2)$$

If i is least possible in (1.2), then x, y double intersperses u from $u_i and x_j, y_k$ respectively.

A Fibonacci sequence is a sequence $s = (s_1, s_2, ...)$ of positive real numbers satisfying

$$s_m = s_{m-1} + s_{m-2} \qquad \text{for } m \geq 3 \qquad (1.3)$$

The classical Fibonacci sequence F_m is given by (1.3) using initial terms

$$F_1 = s_1 = 1 \ \ \& \ \ F_2 = s_2 = 1$$

We shall prove that each 3-tuple of distinct Fibonacci sequences eventually intersperse or doubly intersperse. Double interspersions yield pairs of ordinary interspersions[2].

2. *Interspersion of sequences :*

Lemma 1 : Suppose that u, x and y are Fibonacci sequences satisfying

$$u_i < x_j < y_k < u_{i+1} < x_{j+1} < y_{k+1} < u_{i+2}$$

for some i, j & k.

Then , $\qquad u_{i+h} < x_{j+h} < u_{i+h+1}$

$$\& \qquad u_{i+h} < y_{k+h} < u_{i+h+1} \qquad (2.1)$$

Proof : For $h = 0$ and $h = 1$, the inequalities (2.1) hold by hypothesis.

Assume for arbitrary $h \geq 2$ that

$$u_{i+h-2} < x_{j+h-2} < u_{i+h-1}$$

$$\& \qquad u_{i+h-1} < x_{j+h-1} < u_{i+h}$$

This gives

$$u_{i+h-2} + u_{i+h-1} < x_{j+h-2} + x_{j+h-1} < u_{i+h-1} + u_{i+h}$$

Which results in

$$u_{i+h} < x_{j+h} < u_{i+h+1}$$

Similarly,

$$u_{i+h} < y_{k+h} < u_{i+h+1}.$$

Lemma 2: Suppose that *u,x & y* are Fibonacci sequences satisfying

$$x_m = u_m \ and \ x_{m+h} = u_{m+h} \quad and$$

$$y_n = u_n \ and \ y_{n+h} = u_{n+h} \quad for\ some$$

$$m \geq 1,\ h \geq 1\ and\ n \geq 1,$$

$$Then \quad x = u \ \ and \ \ y = u\ .$$

Proof : Given that $x_m=u_m$ if $x_{m+1}=u_{m+1}$, then clearly $x = u$.

On the other hand, if $x_{m+1}<u_{m+1}$, then $x_{m+2}=u_m + x_{m+1}<u_m+u_{m+1}=u_{m+2}$

And inductively,

$x_{m+h}<u_{m+h}$ for all $h \geq 1$ which is contrary to the hypothesis.

If $u_{m+1} < u_{m+1}$ then inductively

$u_{m+h} < x_{m+h}$, for all $h \geq 1$, again a contradiction.

Therefore, $x = u$.

Similarly, $y_n = u_n$ and $y_{n+h} = u_{n+h}$ gives $y = u$.

Lemma 3 : If u is a Fibonacci sequence, then

$$u_n = u_1 F_{n-3} + u_2 F_{n-2} \qquad for\ all\ n \geq 3,$$

Where $F_0 = 0$.

Proof : $\quad F_1 = u_1 = 1, \quad F_2 = u_2 = 1$

$$\& \quad u_m = u_{m-1} + u_{m-2}$$

By induction

Let us suppose that

$$u_{m-1} = u_1F_{m-4} + u_2F_{m-3}$$

$$\& \quad u_{m-2} = u_1F_{m-5} + u_2F_{m-4}$$

$$\begin{aligned} u_m &= u_{m-1} + u_{m-2} \\ &= u_1F_{m-4} + u_2F_{m-3} + u_1F_{m-5} + u_2F_{m-4} \\ &= u_1(F_{m-4} + F_{m-5}) + u_2(F_{m-3} + F_{m-4}) \\ &= u_1F_{m-3} + u_2F_{m-2} \end{aligned}$$

Which proves the required result for general n.

Theorem : Distinct positive Fibonacci sequences u,x and y eventually doubly intersperse if there exist $a \geq 1$, $b \geq 1$ *and* $c \geq 1$ such that

$$x_b + y_c - 2u_a = \tau(2u_{a+1} - x_{b+1} - y_{c+1}) \quad (2.2)$$

Where τ denotes the golden ratio, $(1+\sqrt{5})/2$

Otherwise, eventually intersperse.

Proof : Suppose that u,x and y are distinct positive Fibonacci sequences i, j & k satisfy

$$u_{i-1} < u_i < x_j < y_k$$

This can be further divided into two sub cases: (I) $u_{i-1} < u_i < x_j$ and (II) $u_{i-1} < u_i < y_k$

In each of these sub case the following cases are exhaustive : (for *I*)

Case 1: $u_i < x_j < u_{i+1} < x_{j+1} < u_{i+2}$

Case 2: $u_i < x_j < x_{j+1} < u_{i+1}$

Case 3: $u_i < x_j < u_{i+1} < u_{i+2} \leq x_{j+1}$

In case 1, the sequences and eventually intersperse by lemma 1.

In case 3, there are two sub cases :

$$x_{j-1} \leq u_i < x_j < u_{i+1} < u_{i+2} \leq x_{j+1} \quad (2.3)$$

$$u_i < x_{j-1} < x_j < u_{i+1} < u_{i+2} \leq x_{j+1} \quad (2.4)$$

Equation (2.3) yields a contradiction.

By equation (2.4) $u_{i+1} < x_{j+1} < u_{i+3}$

Now, if $u_{i+3} < u_{j+2}$, then u and x eventually intersperse by lemma 1.

On the other hand, if

$$u_{i+1} < x_{j+1} < x_{j+2} \leq u_{i+3}$$

Then by shifting indices, we have

$$u_i < x_j < x_{j+1} \leq u_{i+2}$$

Which is covered in case 1.

Now case 2, breaks into two cases:

Case 2.1 $x_{j+2} < u_{i+2}$

Case 2.2 $u_{j+2} \geq u_{i+2}$

In case 2.1, we have

$u_{i-1} \leq u_i < x_j$ *and* $u_i < x_{j+1}$, so that

$$u_{i-1} + u_i < x_j + x_{j+1}$$

i.e., $u_{i+1} < x_{j+2}$

Which gives

$$u_i < x_j < x_{j+1} \leq u_{i+1} < x_{j+2} < u_{i+2}$$

Which implies

$$u_{i+1} < x_{j+2} < u_{i+2} < x_{j+3} < u_{i+3}$$

So that u and x eventually intersperse by lemma 1.

In the remaining case that no such h exists, we have

$$u_i < x_j < x_{j+1} \leq u_{i+1} < u_{i+2} \leq x_{j+2} < x_{j+3} \leq u_{i+3} < u_{i+4} \leq \ldots \ldots \ldots \quad (2.5)$$

Although (2.5) shows $x_{j+h} < u_{i+h}$ for infinitely many h, at most one of these is $x_{j+h} = u_{i+h}$

by lemma 2. Likewise, at most one of the inequalities $u_{i+m} \leq x_{j+m}$ in (2.5) is actually $u_{i+m} = x_{j+m}$

Therefore, in (2.5) we can assume i large enough that

$$u_i < x_j < x_{j+1} < u_{i+1} < u_{i+2} < x_{j+2} < x_{j+3} < u_{i+3} < u_{i+4} < \ldots\ldots\ldots \quad (2.6)$$

which is to say that u and x eventually doubly intersperse. From (2.6) we extract two families of inequalities :

$$u_i < x_j$$

$$u_i + u_{i+1} = u_{i+2} < x_{j+2} = x_j + x_{j+1}$$

$$2u_i + 3u_{i+1} = u_{i+4} < x_{j+4} = 2x_j + 3x_{j+1}$$

$$\vdots$$

$$F_n u_i + F_{n+1} u_{i+1} < F_n x_j + F_{n+1} x_{j+1} \quad \text{for } n = 1, 3, 5, \ldots\ldots\ldots \quad (2.7)$$

and $x_{j+1} < u_{i+1}$

$$x_j + 2x_{j+1} = x_{j+3} < u_{i+3} = u_i + 2u_{i+1}$$

$$3x_j + 5x_{j+1} = x_{j+5} < u_{i+5} = 3u_i + 5u_{i+1}$$

$$\vdots$$

$$F_n x_j + F_{n+1} x_{j+1} < F_n u_i + F_{n+1} u_{i+1} \quad \text{for } n = 2, 4, 6, \ldots\ldots\ldots \quad (2.8)$$

Similarly, if we consider the case II

$u_{i-1} < u_i < y_k$

we arrive at the result that either u and y eventually intersperse or we can assume i large enough that –

$$u_i < y_k < y_{k+1} < u_{i+1} < u_{i+2} < y_{k+2} < y_{k+3} < u_{i+3} < u_{i+4} < \ldots\ldots\ldots$$

which is to say that u and y eventually doubly intersperse and this result in two inequalities :

$$F_n u_i + F_{n+1} u_{i+1} < F_n y_k + F_n y_{k+1} \quad \text{for } n = 1, 3, 5, \ldots\ldots \quad (2.9)$$

and

$$F_n y_k + F_{n+1} y_{k+1} < F_n u_i + F_{n+1} u_{i+1} \quad \text{for } n = 2, 4, 6, \ldots\ldots \quad (2.10)$$

Eq. (2.7) & (2.9) gives -

$$2F_n u_i + 2F_{n+1} u_{i+1} < F_n (x_j + y_k) + F_{n+1}(x_{j+1} + y_{k+1}) \quad \text{for } n=1, 3, 5, \ldots\ldots$$

Through eq. (2.8) and (2.10) we arrive at -

$$F_n (x_j + y_k) + F_{n+1}(x_{j+1} + y_{k+1}) < 2F_n u_i + 2F_{n+1} u_{i+1} \quad \text{for } n= 2, 4, 6, \ldots\ldots$$

These inequalities are equivalent to

$$\frac{F_{n+1}}{F_n} < \frac{x_j + y_k - 2u_i}{2u_{i+1} - x_{j+1} - y_{k+1}} \quad \text{for } n=1,3,5,\ldots\ldots$$

$$\frac{F_{n+1}}{F_n} > \frac{x_i + y_k - 2u_i}{2u_{i+1} - x_{j+1} - y_{k+1}} \quad \text{for } n=2,4,6,\ldots\ldots$$

Taking limit $n \to \infty$ the pair of inequality reduce to

$$\tau = \frac{x_b + y_c - 2u_a}{2u_{a+1} - x_{b+1} - y_{c+1}}$$

Proof of the Converse: Let us suppose that u,x and y are distinct positive Fibonacci sequences for which there exist $a \geq 1, b \geq 1$ *and* $c \geq 1$ such that

$$x_b + y_c - 2u_a = \tau(2u_{a+1} - x_{b+1} - y_{c+1})$$

this means that $\tau = \dfrac{x_b + y_c - 2u_a}{2u_{a+1} - x_{b+1} - y_{c+1}}$

Which gives $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \dfrac{x_b + y_c - 2u_a}{2u_{a+1} - x_{b+1} - y_{c+1}}$

Hence there exist i, j and k such that

$$\left| \frac{F_{n+1}}{F_n} - \frac{x_j + y_k - 2u_i}{2u_{i+1} - x_{j+1} - y_{k+1}} \right| < \varepsilon$$

which gives two inequalities

$$\frac{F_{n+1}}{F_n} < \frac{x_j + y_k - 2u_i}{2u_{i+1} - x_{j+1} - y_{k+1}}$$

& $$\frac{F_{n+1}}{F_n} > \frac{x_j + y_k - 2u_i}{2u_{i+1} - x_{j+1} - y_{k+1}}$$

or, $2F_{n+1}u_{i+1} + 2F_n u_i < F_n(x_j + y_k) + F_{n+1}(x_{j+1} + y_{k+1})$

& $2F_{n+1}u_{n+1} + 2F_n u_i > F_n(x_j + y_k) + F_{n+1}(x_{j+1} + y_{k+1})$

or, $F_{n+1}u_{i+1} + F_n u_i < F_n x_j + F_{n+1}x_{j+1}$ (2.12)

$F_{n+1}u_{i+1} + F_n u_i < F_n x_j + F_{n+1}x_{j+1}$ (2.13)

$F_{n+1}u_{i+1} + F_n u_i < F_n y_k + F_{n+1}y_{k+1}$ (2.14)

$F_{n+1}u_{i+1} + F_n u_i > F_n y_k + F_{n+1}y_{k+1}$ (2.15)

If we put n = 1, 3, 5,………….in (2.12) and (2.14) and n=2, 4, 6, ……….. in (2.13) and (2.15), Then combining the result with the condition $x_{j+m} < y_{k+m-1}$ for m = 1, 3, 5, ……..
We arrive at

$$u_i < x_j < x_{j+1} < y_k < y_{k+1} < u_{i+1} < u_{i+2} < x_{j+2}$$
$$< x_{j+3} < y_{k+2} < y_{k+3} < u_{i+3} < u_{i+4} < \ldots\ldots.$$

Hence, distinct positive Fibonacci sequences eventually doubly intersperse.

Corollary : Suppose that *u*, *x* and *y* eventually doubly intersperse and that *a*,*b* and *c* are indices for which equation (2.1) hold, then

$$2u_{a+1} - x_{b+1} - y_{c+1} = (F_{n+1} - \tau F_n)$$
$(2u_a - x_b - y_c)$ for n = 0, 1, 2,….. (2.16)

Proof: The equation (2.2) is equivalent to

$$2u_{a+1} - x_{b+1} - y_{c+1} = \frac{1}{\tau}(x_b + y_c - 2u_a)$$
$$= (F_2 - \tau F_1)(2u_a - x_b - y_c)$$

Then

$$\begin{aligned} 2u_{a+2} - x_{b+2} - y_{c+2} &= 2(u_{a+1} + u_a) - (x_{b+1} + x_b) - (y_{c+1} + y_c) \\ &= (2u_{a+1} - x_{b+1} - y_{c+1}) + (2u_a - x_b - y_c) \\ &= (1-\tau)(2u_a - x_b - y_c) + (2u_a - x_b - y_c) \\ &= (1-\tau+1)(2u_a - x_b - y_c) \\ &= (2-\tau)(2u_a - x_b - y_c) \\ &= (F_3 - \tau F_2)(2u_a - x_b - y_c) \end{aligned}$$

so it is true for n = 0, 1, 2.
let it be true for n = k

$$2u_{a+k} - x_{b+k} - y_{c+k} = (F_{k+1} - \tau F_k)(2u_a - x_b - y_c)$$

Now, for n = k+1

$$\begin{aligned} &2u_{a+k+1} - x_{b+k+1} - y_{c+k+1} \\ &= 2(u_{a+k} + u_{a+k-1}) - x_{b+k} - x_{b+k-1} - y_{c+k} - y_{c+k-1} \\ &= (2u_{a+k} - x_{b+k} - y_{c+k}) + (2u_{a+k-1} - x_{b+k-1} - y_{c+k-1}) \\ &= (F_{k+1} - \tau F_k)(2u_a - x_b - y_c) + (F_k - \tau F_{k-1})(2u_a - x_b - y_c) \\ &= (F_{k+1} + F_k) - \tau(F_k + F_{k-1})(2u_a - x_b - y_c) \\ &= (F_{k+2} - \tau F_{k+1})(2u_a - x_b - y_c) \end{aligned}$$

So the result is true for n = k+1
Hence it is true for every n by Mathematical induction.

Corollary : Suppose that *u*, *x* and *y* eventually doubly intersperse and that *a*, *b* and *c* are indices for which equation (2.2) holds, then

$$x_{b+n}+y_{c+n}=(F_{n+1}-\tau F_n)(x_b+y_c)$$
$$+2F_n(u_{a+1}+(\tau-1)u_a) \text{ for n = 0, 1, 2,....} \quad (2.17)$$

Proof : The equation (2.2) can be written as

$$x_{b+1}+y_{c+1}=2u_{a+1}-\frac{1}{\tau}(x_b+y_c-2u_a)$$
$$=2u_{a+1}-\tau^{-1}(x_b+y_c-2u_a)$$
$$=2u_{a+1}+(1-\tau)(x_b+y_c-2u_a)$$
$$=(1-\tau)(x_b+y_c)+2(u_{a+1}+(\tau-1)u_a)$$
$$=(F_2-\tau F_1)(x_b+y_c)+2F_1(u_{a+1}+(\tau-1)u_a)$$

Similarly,

$$x_{b+2}+y_{c+2}=(F_3-\tau F_2)(x_b+y_c)$$
$$+2F_2(u_{a+1}+(\tau-1)u_a)$$

Hence by Mathematical Induction the result follows.

Corollary: Suppose that $u=(u_1,u_2,\ldots)$, $x=(x_1,x_2,\ldots)$ & $y=(y_1,y_2,\ldots)$, are positive Fibonacci Sequences satisfying

$$u_1<x_1<x_2<u_2<u_3<x_3<x_4<u_4<u_5<\ldots\ldots$$

and

$$u_1<y_1<y_2<u_2<u_3<y_3<y_4<u_4<u_5<\ldots\ldots$$

Then,

$$x_1+y_1<(2-\tau)u_1+(\tau-1)u_2$$

Proof : Taking n = 1, a = 1, b = 1, c = 1 in (2.17) gives

$$x_2+y_2=(F_2-\tau F_1)(x_1+y_1)+2F_1(u_2+(\tau-1)u_1)$$
$$=(1-\tau)(x_1+y_1)+(u_2+(\tau-1)u_1)>(x_1+y_1)$$

This proves

$$(x_1+y_1)<\frac{u_2+(\tau-1)u_1}{\tau}=(2-\tau)u_1+(\tau-1)u_2$$

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