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Banach contraction principle on cone generalized metric space

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Abstract

In this paper, we introduce the notion of cone generalized metric space and discuss some properties of convergence of sequences. Further we prove a fixed point theorem for contractive mapping in cone generalized metric space.

Keywords and phrases: fixed point; Contraction mapping principle; Cone generalized metric space; Cone metric space; Cone rectangular metric space.

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1 Introduction

Huang and Zhang[4] introduced the concept of cone metric space and proved some fixed point theorems for contractive mappings in normal cone metric space. Later, several other authors[3,5,6,7] studied the existence of fixed points and common fixed points of mappings satisfying a contractive type condition on a normal cone metric space. Recently Akbar *et al.*[8] introduced the concept of cone rectangular metric spaces and proved Banach contraction mapping principle in cone rectangular metric space.

In the paper we introduce cone generalized metric spaces and prove Banach contraction mapping principle in a complete normal cone generalized metric space.

2 Preliminaries :

The following notions have been used to prove the main result.

Definition 2.1: Let E be a real Banach Space. A subset P of E is called cone[4] if and only if

(i) P is closed, non empty and $P \neq \{0\}$.
(ii) $0 \leq a, b \in R$ and $x, y \in P \Rightarrow ax + by \in P$.
(iii) $P \cap (-P) = \{0\}$.

Definition 2.2: The partial ordering[4] $\leq$ with respect to $P \subseteq E$ is defined by $x \leq y$ if and only if $y - x \in P$. $x < y$ shows that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in int(P)$, int(P) denotes the interior of P.

Definition 2.3: A cone P is called normal[4] if there is a number $k \geq 1$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Rightarrow \|x\| \leq k \|y\|.$$

The least positive number k satisfying the above inequality is called the normal constant of P.

In this paper we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.4: Let X be a non empty set. Suppose that the mapping $\rho : X \times X \rightarrow E$ satisfies:

(i) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.
(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, w) + \rho(w, u) + \rho(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u \in X - \{x, y\}$.

Then ρ is called a cone pentagonal metric[9] on X and (X, ρ) is called a cone pentagonal metric space.

3 Fixed point theorem :

In this section we shell define generalized cone metric space and prove a fixed point theorem of contractive mapping.

Definition 3.1 Let X be a non empty set. Suppose that the mapping $d: X \times X \rightarrow E$ satisfies:

(d_1) $0 < d(x, y)$ for all x, y X and $d(x, y) = 0$ if and only if $x = y$.
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(d_3) $d(x_1, x_k) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots\ldots\ldots + d(x_{k-1}, x_k)$ for all $x_2, x_3, x_4 \ldots\ldots x_{k-1} \in X - \{x_1, x_k\}$[Generalized property].

Then d is called a cone generalized metric on X and (X, d) is called a cone generalized metric space.

Definition 3.2 Let $\{x_n\}$ be a sequence in generalized metric space (X, d) and $x \in X$. If for every $c \in E$, with $0 << c$ there exist $n_0 \in N$ and that for all $n > n_0$, $d(x_n, x) << c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\rightarrow\infty} x_n = x$ or $x_n \rightarrow x$, as $n \rightarrow \propto$.

Definition 3.3 If for every $c \in E$, with $0 << c$ there exist $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) << c$, then $\{x_n\}$is called a Cauchy sequence in generalized metric space (X, d).

Definition 3.4 If every Cauchy sequence is convergent in generalized metric space (X, d), then (X, d) is called a complete cone generalized metric space.

Theorem 3.1 Every cone rectangular metric space and so cone metric space is cone generalized metric space.

Proof: Since every cone metric is cone rectangular metric and every cone rectangular metric is cone generalized metric so the proof is obvious.

The converse of the above theorem is not necessarily true as it can be seen from the following example.

Example 3.1 Let $X = N$, $E = R^2$ and $P = \{(x, y): x, y \geq 0\}$. Define $d: X \times X \to E$ as follows:
$d(x, y) = (0, 0)$ if $x = y$; $d(x, y) = (6, 12)$ if x and y are in $\{2, 3\}$, $x \neq y$; $d(x, y) = (2, 4)$ if x and y cannot both at a time in $\{2, 3\}$, $x \neq y$.

Then (X, d) is a cone generalized (or rectangular) metric space but not a cone metric space because it lacks the triangular property:

$(6, 12) = d(2, 3) \;\; d(2, 4) + d(4, 3) = (2, 4) + (2, 4) = (4, 8)$ as $(6, 12) - (4, 8) = (2, 4) \in P$.

Example 3.2 Let $X = \{1, 2, 3, 4\}$, $E = R^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a normal cone in E. Define $d: X \times X \to E$ as follows:
$d(1, 2) = d(2, 1) = (4, 8)$.
$d(1, 3) = d(3, 1) = d(3, 4) = d(4, 3) = d(2, 4) = d(4, 2) = (1, 2)$.
$d(1, 5) = d(5, 1) = d(2, 5) = d(5, 2) = d(3, 5) = d(5, 3) = d(4, 5) = d(5, 4) = (3, 6)$.

Then (X, d) is a cone generalized metric space but not a cone rectangular metric space because it lacks the rectangular property:

$(4, 8) = d(1, 2) \;\; d(1, 3) + d(3, 4) + d(4, 2) = (1, 2) + (1, 2) + (1, 2) = (3, 6)$
as $(4, 8) - (3, 6) = (1, 2) \in P$.

Before proving our main theorem we present two theorems whose proofs are similar to Huang and Zhang [4, lemmas 1 and 4].

Theorem 3.2 Let (X, d) be a generalized cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

Theorem 3.3 Let (X, d) be a generalized cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $\|d(x_n, x_{n+m})\| \to 0$ as $n \to \infty$.

The main theorem of this paper is as follows.

Theorem 3.4 Let (X, d) be a generalized cone metric space and P be a normal cone with normal constant k and the mapping $T: X \to X$ satisfies the contractive condition $d(Tx, Ty) \leq \lambda\, d(x, y)$ for all $x \in X$, where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$. Define a sequence of points in X as follows,
$x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \ldots\ldots\ldots, x_{n+1} = Tx_n = T^{n+1}x_0$.
We can suppose that x_0 is not a periodic point, if $x_n = x_0$ then,
$d(x_0, Tx_0) = d(x_n, Tx_n) = d(T^nx_0, T^{n+1}x_0) \leq \lambda d(T^{n-1}x_0, T^nx_0) \leq \lambda^2 d(T^{n-2}x_0, T^{n-1}x_0) \leq \ldots\ldots\ldots\ldots \leq \lambda^n d(x_0, Tx_0)$.
This shows that $[\lambda^n - 1]\, d(x_0, Tx_0) \in P$.

It again implies that $\left[\dfrac{\lambda^n - 1}{1 - \lambda^n}\right] d(x_0, Tx_0) \in P$.

Thus $-d(x_0, Tx_0) \in P$ and $d(x_0, Tx_0) = 0$ *i. e.*

x_0 is the fixed point of T.

Now we suppose that $x_m \neq x_n$ $\forall$ m, n $\in$ N. Using generalized property for all y $\in$ X, we have

$$d(y, T^{k-1}y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + \ldots\ldots + d(T^{k-2}y, T^{k-1}y)$$
$$\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \ldots\ldots + \lambda^{k-2} d(y, Ty)$$
$$\leq \sum_{i=0}^{k-2} \lambda^i d(y, Ty)$$

Similarly, $d(y, T^{2k-3}y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + \ldots\ldots + d(T^{2k-4}y, T^{2k-3}y)$

$$\leq \sum_{i=0}^{2k-4} \lambda^i d(y, Ty) = \sum_{i=0}^{2(k-2)} \lambda^i d(y, Ty)$$

So by induction, we obtain for each r = 1, 2, 3……….

$$d(y, T^{r(k-2)+1}y) \leq \sum_{i=0}^{r(k-2)} \lambda^i d(y, Ty) \qquad (1)$$

Again for all y $\in$ X,

$$d(y, T^ky) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + \ldots\ldots + d(T^{k-3}y, T^{k-2}y) + d(T^{k-2}y, T^ky)$$
$$\leq d(y, Ty) + d(y, Ty) + \lambda^2 d(y, Ty) + \ldots\ldots + \lambda^{k-3} d(y, Ty) + \lambda^{k-2} d(y, T^2y)$$
$$\leq \sum_{i=0}^{k-3} \lambda^i d(y, Ty) + \lambda^{k-2} d(y, T^2y)$$

Similarly,

$$d(y, T^{2k-2}y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + \ldots\ldots + d(T^{k-3}y, T^{k-2}y) + d(T^{k-2}y, T^{2k-2}y)$$
$$\leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + \ldots\ldots + d(T^{k-3}y, T^{k-2}y) + d(T^{k-2}y, T^{k-1}y) + \ldots\ldots + d(T^{2k-5}y, T^{2k-4}y) + d(T^{2k-4}y, T^{2k-2}y)$$
$$\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \ldots\ldots + \lambda^{k-3} d(y, Ty) + \lambda^{k-2} d(y, Ty) + \ldots\ldots + \lambda^{2k-5} d(y, Ty) + \lambda^{2k-4} d(y, T^2y)$$
$$\leq \sum_{i=0}^{2k-5} \lambda^i d(y, Ty) + \lambda^{2(k-2)} d(y, T^2y).$$

So by induction, we obtain for each r = 1, 2, 3……..

$$d(y, T^{r(k-2)+2}y) \leq \sum_{i=0}^{r(k-2)-1} \lambda^i d(y, Ty) + \lambda^{r(k-2)} d(y, T^2y) \qquad (2)$$

Continuing this process upto k-2 times we get

$$d(y, T^{r(k-2)+k-2}y) \leq \sum_{i=0}^{r(k-2)-1} \lambda^i d(y, Ty) + \lambda^{r(k-2)} d(y, T^{k-2}y) \qquad (3)$$

Using inequality (1) for r = 1, 2, 3…………

$$d(T^nx_0, T^{n+r(k-2)+1}x_0) \leq \lambda^n d(Tx_0, T^{r(k-2)+1}x_0)$$
$$\leq \lambda^n \sum_{i=0}^{r(k-2)} \lambda^i d(x_0, Tx_0)$$
$$\leq \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + \ldots\ldots + d(x_0, T^{k-2}x_0)] \qquad (4)$$

Similarly for r = 1, 2, 3……., inequality (2) implies that

$$d(T^nx_0, T^{n+r(k-2)+2}x_0) \leq \lambda^n d(x_0, T^{r(k-2)+2}x_0)$$
$$\leq \lambda^n [\sum_{i=0}^{r(k-2)-1} \lambda^i d(x_0, Tx_0) + \lambda^{r(k-2)} d(x_0, T^2x_0)]$$
$$\leq \lambda^n [\sum_{i=0}^{r(k-2)} \lambda^i [d(x_0, Tx_0) + d(x_0, T^2x_0) + \ldots\ldots + d(x_0, T^{k-2}x_0)]$$
$$\leq \frac{\lambda^n (1-\lambda^{r(k-2)-1})}{1-\lambda} [d(x_0, Tx_0) +$$

$d(x_0, T^2x_0) + \ldots\ldots\ldots + d(x_0, T^{k-2}x_0)]$

$$\leq \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + \ldots \ldots\ldots + d(x_0, T^{k-2}x_0)] \qquad (5)$$

Continuing this process upto k-2 times, inequality (3) implies that

$d(T^nx_0, T^{n + r(k-2) + k - 2} x_0) \leq \lambda^n d(x_0, T^{r(k-2) + k - 2}x_0)$

$$\leq \lambda^n [\sum_{i=0}^{r(k-2)-1} \lambda^i d(x_0, Tx_0) + \lambda^{r(k-2)} d(x_0, T^{k-2}x_0)]$$

$$\leq \lambda^n \sum_{i=0}^{r(k-2)} \lambda^i [d(x_0,Tx_0) + d(x_0, T^2x_0) + \ldots\ldots\ldots + d(x_0, T^{k-2}x_0)]$$

$$\leq \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + \ldots \ldots\ldots + d(x_0, T^{k-2}x_0)] \qquad (6)$$

Thus by inequality (4), (5) and (6) we have,

$$d(T^nx_0, T^{n+m}x_0) \leq \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + \ldots\ldots\ldots + d(x_0, T^{k-2}x_0)]$$

Since P is a normal cone with normal constant k, therefore,

$$\| d(T^nx_0, T^{n+m}x_0) \| \leq \frac{\lambda^n k}{1-\lambda} \|[d(x_0, Tx_0) + d(x_0, T^2x_0) + \ldots\ldots\ldots + d(x_0, T^{k-2}x_0)]\|$$

i.e. $\| d(x_n, x_m) | \to 0$ as $n \to \infty$.

Therefore by theorem 3.3, sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $x_n \to z$.

Again by theorem 3.2, we have $\| d(T^nx_0, z) \| \to 0$ as $n \to \infty$. Since $x_n \neq x_m$ for $n \neq m$, therefore by generalized property, we have

$d(Tz, z) \leq d(Tz, T^nx_0) + d(T^nx_0, T^{n+1}x_0) + d(T^{n+1}x_0, T^{n+2}x_0) + \ldots\ldots + d(T^{n+k-4}x_0, T^{n+k-5}x_0) + d(T^{n+k-5}x_0, z)$

$\leq \lambda d(z, T^{n-1}x_0) + \lambda^n d(x_0, Tx_0) + \lambda^{n+1} d(x_0, Tx_0) + \ldots\ldots\ldots + \lambda^{n+k-4}d(x_0, Tx_0) + d(T^{n+k-5}x_0, z)$.

Thus we have

$\| d(Tz, z) \| \leq k [\lambda\| d(z, T^{n-1}x_0) \| + \lambda^n \| d(x_0, Tx_0) \| + \lambda^{n+1} \| d(x_0, Tx_0) \| + \ldots\ldots\ldots + \lambda^{n+k-4} \| d(x_0, Tx_0) \| + \| d(T^{n+k-5}x_0, z)\|]$.

Letting $n \to \infty$, we have, $\| d(Tz, z) \| = 0$ *i.e.* $Tz = z$.

Hence z is a fixed point of T.

Now we show that z is unique. For suppose z' be another fixed point of T such that $Tz' = z'$.

$d(z, z') = d(Tz, Tz') \leq \lambda d(z, z')$.

Hence $z = z'$.

This completes the proof of the theorem.

Remark 3.1 In example 3.2, (X, d) is a complete cone generalized metric space but not a complete cone regular metric space because it lacks the rectangular property.

Define a mapping $T: X \to X$ as follows:

$T(x) = 4$ if $x \neq 5$ and $T(x) = 2$ if $x = 5$.

Note that $d(T(1), T(2)) = d(T(1), T(3)) = d(T(1), T(4)) = d(T(2), T(3)) = d(T(2), T(4)) = d(T(3), T(4)) = 0$.

And in all other cases $d(T(x), T(y)) = (1, 2)$, $d(x, y) = (3, 6)$.

Hence, for $\lambda = 1/3$, all conditions of theorem (3.4) are satisfied and 4 is a unique fixed point of T.

Remark 3.2 In the example 3.2 (resp.

3.1), results of AZAM, ARSHAD and BEG[8] (resp. HUANG and ZHANG[4]) are not applicable to obtained the fixed point of the mapping T on X. Since (X, d) is not a cone rectangular metric space (resp. cone metric space).

References

1. K. Deimling; Nonlinear Functional Analysis, Springer-Verlag, (1985).
2. B. E. Rhoades; A comparison of various definition of contractive mappings, *Trans. Amer. Math. Soc., 266,* 257 - 290 (1977).
3. M. Abbas and G. Jungck; Common fixed point results for non commuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl. 341,* 416-420 (2008).
4. L. G. Huang and X. Zhang.; Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. App., 332* (2), 1468 - 1476 (2007).
5. D. Hie and V. Rakocevic; Common fixed points for maps on cone metric space, *J. Math. Anal. Appl. 341,* 876 - 882 (2008).
6. S. Rezapour and R. Hamlbarani; Some notes on paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. App., 345*(2), 719 - 724 (2008).
7. P. Vetro; Comman fixed points in cone metric spaces, *Rend. Circ. Mat. Palermo 56*(3), 464 - 468 (2007).
8. Akbar Azam, Muhammad Arshad and Ismat Beg; Banach contraction principal on cone rectangular metric spaces, *Appli. Anal. and discrete mathematics,* **3**(2), 236-241 (2009).