

Derivation alternator rings

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Abstract

In this paper we prove that a 2-divisible derivation alternator ring without nonzero nilpotent elements is right alternative.

Introduction

In^{4,5} Kleinfeld defined two different generalizations of alternative rings, and for each of these generalizations he showed the simple rings with idempotent $e \neq 1$ to be alternative. Both of these generalizations defined by Kleinfeld are contained in the variety of derivation alternator rings. In³ Hentzel and Smith extended Kleinfeld's results to simple derivation alternator rings with idempotent $e \neq 1$. Hentzel, Hogben and Smith² established some identities of flexible derivation alternator rings. Using these identities they proved that derivation alternator rings without nonzero nilpotent elements are alternative. Without using their method, in a simple way we prove the same result in this paper. We give an example of a derivation alternator ring which is not right alternative. In this paper, we first

show that a 2-divisible derivation alternator ring is power-associative. Also we prove that a 2-divisible derivation alternator ring without nonzero nilpotent elements is right alternative.

Preliminaries :

A Ring R is defined to be right alternative if $(y, x, x) = 0$, where the associator $(x, y, z) = (xy)z - x(yz)$ for all x, y, z in R . It is alternative if $(x, x, y) = 0$. The commutator is defined as $(x, y) = xy - yx$. A 2-divisible non-associative ring R is called a derivation alternator ring if it satisfies the following identities :

$$(x, x, x) = 0, \quad (1)$$

$$(yz, x, x) = y(z, x, x) + (y, x, x)z \quad (2)$$

$$\text{and } (x, x, yz) = y(x, x, z) + (x, x, y)z. \quad (3)$$

It is immediate from (2), (3) and linearized (1) that such rings also satisfy

$$(x, yz, x) = y(x, z, x) + (x, y, x)z. \quad (4)$$

A linear mapping D from a ring to itself is called a derivation provided $D(xy)=D(x)y + xD(y)$. Using the standard notation for the commutator, for any derivation D we have $D([x, y]) = [D(x), y] + [x, D(y)]$. We use the Teichmuller identity

$$f(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0. \quad (5)$$

In fact, in conjunction with (1) any two of the three identities (2), (3) and (4) will imply the third. Since (2), (3) and (4) can be summarized by simply saying that the alternators of such rings are derivation maps, we refer to these rings as derivation alternator rings. Throughout this paper R will denote a 2-divisible derivation alternator ring.

We employ the following identity:

$$(xoy, z) + (yoz, x) + (zox, y) = 0. \quad (6)$$

Where $xoy = xy+yx$. This identity follows from linearized (1), since in any ring $(xoy,z) + (yoz,x) + (zox,y) = (x,y,z) + (y,z,x) + (z,x,y) + (x,z,y) + (z,y,x) + (y,x,z)$. (7)

As in⁴, the identity (5) together with (4), (3) and (1) gives

$$\begin{aligned} (x^2, y, x) &= (x, xy, x) - (x, x, yx) + x(x, y, x) + (x, x, y)x \\ &= x(x, y, x) - (x, x, y)x + x(x, y, x) + (x, x, y)x \\ &= 2x(x, y, x). \end{aligned}$$

Thus we have shown $(x^2, y, x)=2x(x, y, x)$. (8)

We note that when formed from a derivation alternator ring the standard opposite ring is like wise a derivation alternator ring. Thus by going to the opposite ring, (8) becomes $(x, y, x^2) = 2(x, y, x)x$. (9)

Since for a derivation alternator ring R , $D(y) = (x, x, y)$ is by (3) a derivation for each fixed $x \in R$, this implies $(x, x, [y, z]) = [(x, x, y), z] + [y, (x, x, z)]$. (10)

A non-associative ring is power-associative if each of its elements generates an associative subring. The following theorem shows that derivation alternator rings have this property.

Theorem 1: A 2-divisible derivation alternator ring is power-associative.

Proof: Let R be a 2-divisible derivation alternator ring and $x \in R$. For $n > 1$

we define recursively $x^n = x^{n-1}x$ and show that $x^n = x^{n-i} x^i$ for $i = 1, 2, \dots, n-1$. From (1), (2) and (3) we have $(x, x, x) = 0$ and $(x^2, x, x) = 0 = (x, x, x^2)$. Hence $x^n = x^{n-i} x^i$ holds for $n = 2, 3, 4$. We now induct on n , assuming $x^k = x^{k-i} x^i$ whenever $k < n$, and consider the case $n > 4$. To show $x^n = x^{n-i} x^i$ for $i = 1, 2, 3, \dots, n-1$, we make yet a second induction, this time on i . Now $x^n = x^{n-1}x$ holds by definition. We assume that $x^n = x^{n-i} x^i$. From linearized (2) and our induction assumptions on n and i , we have $2[x^n - x^{n-(i+1)} x^{i+1}] = (x^{n-(i+1)}, x^i, x) + (x^{n-(i+1)}, x, x^i) = x^{n-(i+2)} [(x, x^i, x) + (x, x, x^i)] + [(x^{n-(i+2)}, x^i, x) + (x^{n-(i+2)}, x, x^i)]x = 0$. Thus $x^n = x^{n-(i+1)} x^{i+1}$ for $n-i \geq 3$. Next (3) and our induction assumption on n imply $(x, x, x^{n-2}) = x^{n-3} (x, x, x) + (x, x, x^{n-3})x = 0$, so that $x^2 x^{n-2} = xx^{n-1}$. But (8) and our induction assumption on n imply $(x^2, x^{n-3}, x) = 2x(x, x^{n-3}, x) = 0$, whence also $x^{n-1}x = x^2 x^{n-2}$. Thus $x^n = x^{n-1}x = x^2 x^{n-2} = xx^{n-1}$, which completes both inductions and there by

the proof of the theorem. \square

To prove that R is flexible, first we prove the following :

Lemma 1: In a derivation alternator ring $(x, y, x)^2 = 0$.

Proof : In theorem 1 it is shown that the derivation alternator rings are power-associative. Hence using (8), (9) and then (4) we have $2x[xo(x, y, x)] = x[(x^2, y, x) + (x, y, x^2)] = (x^2, xy, x) + (x, xy, x^2)$. But using (8), (9) and (4) again we see $(x^2, xy, x) + (x, xy, x^2) = 2xo(x, xy, x) = 2xo(x(x, y, x))$. Thus we arrive at $2x[xo(x, y, x)] = 2xo[x(x, y, x)]$. Since R is 2-divisible and cancelling $x[x(x, y, x)]$, this last equation implies $x[(x, y, x)x] = [x(x, y, x)]x$, or $(x, (x, y, x), x) = 0$. (11)

We can now utilize (11), (4) twice, and (11) again, to obtain $0 = (x, (x, y^2, x), x) = (x, y o (x, y, x), x) = y o (x, (x, y, x), x) + (x, y, x) o (x, y, x) = (x, y, x) o (x, y, x) = 2(x, y, x)^2$. Since R is 2-divisible, this proves $(x, y, x)^2 = 0$. \square

Now we establish some identities of flexible derivation alternator rings.

Lemma 2 : A flexible derivation alternator ring satisfies the identities

$$(x^2, y, z) = xo(x, y, z), \quad (12)$$

$$(y, x^2, z) = xo(y, x, z), \quad (13)$$

$$(y, z, x^2) = xo(y, z, x). \quad (14)$$

Proof : Since $(x, y, x) = 0$, (15)

by assumption, from (8) we see $(x^2, y, x) = 0$. This means that by definition a flexible

derivation alternator ring is non commutative Jordan, and non commutative Jordan rings satisfy (13). A proof of this fact can be found in¹. Next the identity (5), which holds in any ring, says

$$(x^2, y, z) - (x, xy, z) + (x, x, yz) = x(x, y, z) + (x, x, y)z. \quad (16)$$

Like wise (5) implies

$$(zy, x, x) - (z, yx, x) + (z, y, x^2) = z(y, x, x) + (z, y, x)x. \quad (17)$$

Subtracting (17) from (16) and using linearized (15), we arrive at $2(x^2, y, z) + (x, x, yoz) + (z, xoy, x) = xo(x, y, z) + zo(x, x, y)$. Now (3) implies $(x, x, yoz) = yo(x, x, z) + zo(x, x, y)$, which we substitute into the last equation. After cancelling $zo(x, x, y)$, we then have $2(x^2, y, z) + yo(x, x, z) + (z, xoy, x) = xo(x, y, z)$. Next linearizing the already established (13), and utilizing flexibility, we see that $(z, xoy, x) = xo(z, y, x) + yo(z, x, x) = -\{xo(x, y, z) + yo(x, x, z)\}$. Substituting this into our previous equation leads to $2\{(x^2, y, z) - xo(x, y, z)\} = 0$. Since R is 2-divisible, this establishes 2.2.12. Flexibility and (12) clearly implies $(y, z, x^2) = xo(y, z, x)$. This Completes the proof of the lemma. \square

Theorem 2: A 2-divisible derivation alternator ring R without nonzero nilpotent elements is right alternative.

Proof: By lemma 1 we have $(x, y, x)^2 = 0$. So the assumption that there are no nonzero nilpotent elements implies $(x, y, x) = 0$. Then $(y, x, x) + (x, x, y) = 0$, using (1). This implies that $(y, x, x) = -(x, x, y)$. Thus $((y, x, x), x, x) = -((x, x, y), x, x) = ((x, x, x), x, y) = 0$. Using (12), we write $0 = ((y^2, x, x), x, x) =$

$(yo(y, x, x), x, x) = yo((y, x, x), x, x) + (y, x, x)o(y, x, x) = (y, x, x) o(y, x, x) = 2(y, x, x)^2$. Since R is 2-divisible, this proves that $(y, x, x)^2 = 0$. Since R has no non zero nilpotent elements, we obtain $(y, x, x) = 0$. Hence R is right alternative. \square

The right alternativity with flexibility implies that R is alternative.

Corollary: A derivation alternator division ring is alternative.

We give an example of a derivation alternator ring which is not right alternative.

Example: Suppose that the ring R is defined by the following multiplication table together with all finite sums of e, a, b, c, d, h, such that $x + x = 2x \neq 0$.

	e	a	b	c	d	h
e	e	b	0	0	0	0
a	h	c	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	0
h	h + b	0	0	0	0	0

First we observe that $(a, a, a) = a^2a - aa^2 = ca - ac = 0 - 0 = 0$. Therefore, R is third power

associative. Now $(ab, e, e) - a(b, e, e) - (a, e, e)b = (ab.e)e - ab(e^2) - a\{be.e - b.e^2\} - \{ae.e - a.e^2\}b = (0.e)e - 0.e - a\{0.e - b.e\} - \{h.e - a.e\}b = 0 - 0 - a\{0 - 0\} - \{h + b - h\}b = b.b = 0$. Therefore $(ab, e, e) - a(b, e, e) - (a, e, e)b = 0$. Since $(a, e, e) = ae.e - a.e^2 = h.e - ae = h + b - h = b \neq 0$, R is not right alternative. Therefore, R is derivation alternative ring but not right alternative.

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