Ultra Scientist Vol. 25(1)B, 171-179 (2013).

# Heat conduction and I-function of several variables

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(Acceptance Date 18th December, 2012)

## Abstract

In the present paper, we evaluate two integrals involving the I-function of several complex variables defined by Sharma and Ahmed[14] and their applications are made in solving problem on heat conduction by Bhonsle[1], and in establishing two expansion formulae involving the above function.

## 1. Introduction

Singh[3] evaluated some integrals involving kampe de Feriet function and one of them was employed to obtain a solution of a problem in heat conduction given by Bhonsle[1].

Recently, Chandel and Yadava[3] have evaluated certain integrals involving multiple hypergeometric function of Srivastava and Daoust[13] and their application have been shown in solving the same problem on heat conduction.

In continuation of the above study, the present paper is inspired by the frequent requirement of various properties of special function which play a vital role in the study of potential theory heat conduction and other allied problems in quantum mechanics. Appell's functions and the functions related to them have many applications in mathematical physics[6,9,10]. Recently I-function of several complex variables has been defined by Sharma and Ahmed[15] in the following way.

$$I^{o,n\;:(m_1,n_1)\;;---;(m_\gamma n_\gamma)}_{P_i,q_i:R:[P_i',q_i':R'];---;[P_i^{(r)},q_i^{(r)}:R^{(\gamma)}]}\left[\begin{matrix} z_1 \\ \vdots \\ z_\gamma \end{matrix}\middle| \begin{matrix} \left[\left(a_j;\alpha_j',\ldots\alpha_j^{(r)}\right)_{1,n}\right] \\ \left[\left(b_j;\beta_j',\ldots\beta_j^{(r)}\right)_{1,q}\right] \end{matrix}\right.$$

$$[(a_{ji};\alpha_j',---\alpha_{ji}^{(r)})_{n+1,P_i}]:\begin{matrix}[(c_j',\gamma_j')1,n_1],[(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];\ldots \\ [(d_j',\delta_j')1,m_1],[(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];\ldots\end{matrix}$$

$$\left. \begin{matrix} ...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_\gamma}],[(c_{ji(\gamma)}^{(r)},\delta_{ji(\gamma)}^{(r)})_{n_r+1,P_i^{(r)}}], \\ ...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_\gamma}],[(d_{ji(\gamma)}^{(r)},\delta_{ji(\gamma)}^{(r)})_{m_r+1,q^{(r)}}], \end{matrix} \right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{Li}\int_{Lr}\frac{\prod_{j=1}^{m_r}\Gamma(d_j^{(r)}-\delta_j^{(r)}\xi_r)\prod_{j=1}^{n_r}\Gamma(1-c_j^{(r)}+\gamma_j^{(r)}\xi_r)}{\sum_{i^{(r)}=1}^{R^{(r)}}\left[\prod_{j=m_r+1}^{q_i}\Gamma(1-d_{ji}^{(r)}+\delta_{ji}^{(r)}\xi_r)\prod_{j=n_r+1}^{P_i^{(r)}}\Gamma(c_{ji}^{(r)}-\gamma_{ji}^{(r)}\xi_r)\right]}$$

$$\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{r=1}^{k}\alpha_j^{(r)}\xi_r)}{\sum_{i=1}^{R}\left[\prod_{j=n+1}^{P_i}\Gamma(a_{ji}-\sum_{r=1}^{k}\alpha_{ji}^{(r)}\xi_r)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{r=1}^{k}\beta_{ji}^{(r)}\xi_r)\right]}d\xi_1,...d\xi_r \qquad (1.1)$$

An empty product is interpreted as 1, the coefficient $\alpha_j^{(r)}, j=1....P_i, \gamma_j^{(r)}, j=1..P_i^{(r)}$, $\beta_j^{(r)}, j=1....q_i, \delta_j^{(r)}, j=1,...q_i^{(r)}, r=1$ $...R$ *and* $n, m_r, n_r, P_i, q_i, P_i^{(r)}, q_i^{(r)}, m$ are positive integers such that, $0<m<P_i$, $0<m_r<q_i(r)$ and $0<n_r<P_i(r)$, r=1 …n.

The contour $L_r$ in the complex $\xi_r$ - plane is of the mellin-Barnes type which runs from $-\omega\infty$ to $+w\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma(d_j^{(r)}-\delta_j^{(r)}\xi_r),\ j=1...m_r$ are to the right and those of

$\Gamma(1-c_j+\gamma_j^{(r)}\xi_r), j=1....n_r$ *and*

$\Gamma(1-a_j+\sum_{r=1}^{k}\alpha_j^{(r)}\xi_r) \qquad j=1....n$

to the left of $L_r$, the various parameters being so restricted that theses poles are all simple and none of them coincide and with the points $z_r=0$, r=1…n being tacitly excluded, the multiple integral in (1.1) converges absolutely[7,8] if

$$\text{larg } (z_r)1<\frac{1}{2}\pi\ \Delta_r, r=1....n, \qquad (1.2)$$

where,

$$\Delta_r=\sum_{j=1}^{n}\alpha_j^{(r)}-\sum_{j=n+1}^{P_i}\alpha_{ji}^{(r)}+\sum_{j=1}^{n_r}\gamma_j^{(r)}-\sum_{j=n_r+1}^{P_i^{(r)}}\gamma_{ji}^{(r)}$$

$$-\sum_{j=1}^{q_i}\beta_{ji}^{(r)}+\sum_{j=1}^{m_r}\delta_j^{(r)}-\sum_{j=m_r+1}^{q_i}\delta_{ji}^{(r)}>0.$$

r = 1…n.

In the present paper, we shall evaluate certain integrals involving the above I-function and their applications will be made in solving problem on heat conduction given by Bhonsle[1] and in establishing some expansion formulae involving the above function[12].

*2. Formulae required :*

Multiplying both sides of Lebedev equation [8, (u, 16, 1)] by $e^{-z^2} H_{2\nu}(z)$ and using orthogonal property of Hermite polynomials[11], we have,

$$\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z)\, dz = \frac{\sqrt{\pi}\, 2^{(\nu-\rho)}\Gamma(2\rho+1)}{\Gamma(\rho-\upsilon+1)}, \rho = 0,1,2.... \tag{2.1}$$

which will be useful in our investigations.
Another formulae required in our investigation is due to [4].

$$\int_{-1}^{1}(1-z^2)^{\rho-1} P_{\upsilon}^{\mu}(z)dz = \frac{\pi 2^{\mu}\Gamma\left(\rho+\frac{\mu}{2}\right)\Gamma\left(\rho-\frac{\mu}{2}\right)}{\Gamma\left(1+\rho+\frac{\vartheta}{2}\right)\Gamma\left(\rho-\frac{\vartheta}{2}\right)\Gamma\left(\frac{\vartheta}{2}+\frac{\mu}{2}+\frac{1}{2}\right)},$$

$$2R_e(\rho) > R_e(\mu)$$

*3. Integrals :*

Making an appeal to (2.1), we obtain

$$\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\vartheta}(z)\, I^{0,n\,:(m_1,n_1);\ldots;(m_r,n_r)}_{P_i,q_i:R:[P_{i'},q_i';R'];\ldots;[P_i(r),q_i(r):R^{(r)}]}$$

$$\left[\begin{matrix} x_1\, z^{2\alpha} \\ \vdots \\ x_r \end{matrix}\left|\begin{matrix}\left[\left(a_j, \alpha_j' \ldots \alpha_j^{(r)}\right)_{1,n}\right]\left[\left(a_{ji};\alpha_{ji}' \ldots \alpha_{ji}^{(r)}\right)_{n+1,P_i}\right] \\ \left[\left(b_j, \beta'_{ji} \ldots \beta_{ji}^{(r)}\right)_{1,q_i}\right]\end{matrix}\right.\right.$$

$$[(c_j', \gamma_j')_{1,n_1}][(c_{ji}', \gamma_{ji}')_{n_1+1,P_i}];\ldots;[(c_j^{(r)}, \gamma_j^{(r)})_{1,n_r}]$$

$$[(d_j', \delta_j')_{1,m_1}][(d_{ji}', \delta_{ji}')_{m_1+1,q_i}];\ldots;[(d_j^{(r)}, \delta_j^{(r)})_{1,m_r}]$$

$$\left.\begin{matrix}[(c_{ji^{(r)}}^{(r)} \;,\; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,P_i^{(r)}}] \\ [(d_{ji^{(r)}}^{(r)},\, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i(r)}]\end{matrix}\right] dz$$

$$= \sqrt{\pi}\; 2^{2(\upsilon-\zeta)}\; I^{0,n \;:\; (m_1+1,\; n_1);....;(m_r,\; n_r)}_{[p_i,q_i:R]:\;[p_{i'},+1,\;\; q;'+1,\;\; :R'];...;[p_i^{(r)},q_i^{(r)}:R^{(r)}]}$$

$$\left[\begin{matrix} x_1\; z^{-2\alpha} \\ \vdots \\ x_r \end{matrix}\middle| \begin{matrix}(a_j,\alpha_j^{'}...\alpha_j^{(r)}1,n; \quad (-2\rho,2\alpha)\;[(c_j{'},\gamma_j{'})_{1,n_1}]\,[(c_{ji}{'};\gamma_{ji}{'})_{n+1,p_i}] \\ (b_{ji},\beta_{ji}{'}...\beta_{ji}^{(r)})1,q_i : (\upsilon-\rho,\alpha)\;[(d_j{'},\delta_j{'})_{1,m_1}]\,[(d_{ji}{'};\delta_{ji}{'})_{m_1+1,q_i}]\end{matrix}\right.$$

$$\left.\begin{matrix};......[(c_j^{(r)},\gamma_j^{(r)},)_{1,n_r}]\,[(c_{ji_{(r)}}^{(r)},\gamma_{ji_{(r)}}^{(r)})_{n_r+1,\; p_i^{(r)}}] \\ ;......[(d_j^{(r)},\delta_j^{(r)},)_{1,m_r}]\;[(d_{ji_{(r)}}^{(r)},\delta_{ji_{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{matrix}\right] dz \tag{3.1}$$

where,

$$\alpha_i > 0,\; l\arg\; x_i 1 < \frac{1}{2}\pi, \Delta_r, \mathrm{Re}\,[1+2\rho+2\sum_{i=1}^{r}\alpha_i\gamma_i] > 0,$$

$$\Delta_r = \sum_{j=1}^{n}\alpha_j^{(r)} - \sum_{j=n+1}\alpha_{ji}^{(r)} + \sum_{j=1}^{n_r}\gamma_j^{(r)} - \sum_{j=n_r+1}^{p_i(r)}\gamma_{ji}^{(r)} - \sum_{j=1}^{q_i}\beta_{ji}^{(r)} + \sum_{j=1}^{m_r}\delta_j^{(r)} - \sum_{j=m_r+1}^{q_i}\delta_{ji}^{(r)} > 0, \mathrm{r} = 1...\mathrm{n}$$

and, $\gamma_r = min\;\mathrm{Re}\left\{\dfrac{d_j^{(i)}}{\delta_j^{(i)}}\right\}\;(j = 1...m_r)$

Particularly for $\lambda = 0$, the above result reduces to the result due to Bohara and Jain[2] [2,(2.4)]. Similarly an appeal to (2.2), shows that

$$\int_{-1}^{1}(1-z^2)^{\rho-1}P_\nu^\mu(z)\; I^{0,n \;:(m_1,n_1);...;(m_r,n_r)}_{P_i,q_i:R:[P_{i'},q;':R'];...;[P_i(r),q_i(r):R^{(r)}]}$$

$$\left[\begin{matrix} x_1\;(1-z^2)^\alpha \\ \vdots \\ x_n \end{matrix}\middle| \begin{matrix}[(a_j;\alpha_j^{'}...\alpha_j^{(r)})_{1,n}] \quad [a_{ji};\alpha_{ji}{'}....\alpha_{ji}^{(r)})_{n+1,\;p_i}]; \\ (b_j,;\beta_{ji}{'}...\beta_{ji}^{(r)})_{1,q_1}]\end{matrix}\right.$$

$$[(c_j',\gamma_j')_{1,n_1}][(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];\ldots;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}]$$
$$[(d_j',\delta_j')_{1,m_1}][(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];\ldots;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}]$$

$$\left.\begin{matrix}[(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}})_{n_r+1,P_i^{(r)}}]\\ [(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}})_{m_r+1,q_i^{(r)}}]\end{matrix}\right] dz$$

$$= \frac{\pi 2^{\mu}}{\Gamma\left(\frac{\upsilon+\mu}{2}+1\right)\Gamma\frac{(1-\mu-\upsilon)}{2}} I^{0,n\ :\ (m_1+2,\ n_1+2);\ldots;(m_r,n_r)}_{[P_i',q_i':R']:\ [P_i'+2,q_i'+2,R'];\ldots\ldots;[P_i^{(r)},\ q_i^{(r)}:R^{(r)}]}$$

$$\left[\begin{matrix}x_1\\ \vdots\\ x_r\end{matrix}\left|\begin{matrix}[((a_j,\alpha_j'\ldots\alpha_j^{(r)})_{1,n}](1-\rho-\mu/2,\alpha),(1+\rho+\mu/2,\alpha):[(c_j',\gamma_j')_{1,n_1}]\\ [(b_{ji},\beta_{ji}'\ldots\beta_{ji}^{(r)})_{1,q_i}]:(-\rho-\upsilon/2,\alpha),\ (1-\rho+\upsilon/2,\alpha):[(d_j',\delta_j')_{1,m_1}]\end{matrix}\right.\right.$$

$$\left.\begin{matrix}[(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];\ldots;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}]\,[(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}})_{n_r+1,P_i^{(r)}}]\\ [(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];\ldots;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}]\,[(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}})_{m_r+1,q_i^{(r)}}]\end{matrix}\right] dz \qquad (3.2)$$

where, 2Re(ρ) > 2 Re$(\mu),(\rho\pm\mu/2-\sum_{r=1}^{n}\alpha_r\gamma_r)>0$

$\alpha_r>0,|\arg x_r|<\lambda/2\,\Delta_r$ and $\gamma_r$ are as in (3.1), $r=1,\ldots n$

*4. Application in heat conduction:*

Bhonsle[1] has employed Hermite polynomials in solving the partial differential equation.

$$\frac{\partial\phi}{\partial t}=k\frac{\partial^2\phi}{\partial z^2}-k\phi z^2 \qquad (4.1)$$

where $\phi(z,t)$ tends to zero for large of t and when $|z|\to\infty$, this equation related to the problem of heat conduction[5]

$$\frac{\partial\phi}{\partial t}=k\frac{\partial^2\phi}{\partial z^2}-h_1'(\phi-\phi_0) \qquad (4.2)$$

Provided that $\phi_0=0$ and $h_1=kz^2$,

The solution of (4.1) given by Bhosle[1] Rs.

$$\phi(z,t)=\sum_{r=0}^{\infty}\phi,e^{-(1+2r)kt-z^2/2}H_r(z) \qquad (4.3)$$

we shall consider the problem of determining a function $\phi(z,t)$; if $t=0$, then

$$\phi(z,0)=z^{2\rho}e^{-z^2}I^{0,n\ :(m_1,n_1);...;(m_r,n_r)}_{p_i,q_i:R:[p_{i'},q_i';R'];...;[p^{(r)}_{i(r)},q^{(r)}_{i(r)}:R^{(r)}]}\left[\begin{matrix}x_1\ z^{2\alpha}\\ \vdots\\ x_r\end{matrix}\middle|\begin{matrix}[(a_j;\alpha_j'...\alpha_j^{(r)})_{1,n}]\ \ [(a_{ji};\alpha_{ji}'....\alpha_{ji}^{(r)})_{n+1,P_i}];\\ (b_j,;\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}]\end{matrix}\right.$$

$$\left.\begin{matrix}[(c_j',\gamma_j')_{1,n_1}][(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}]\,[(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}})_{n_r+1,P_i^{(r)}}]\\ [(d_j',\delta_j')_{1,m_1}][(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}]\,[(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}})_{m_r+1,q_i^{(r)}}]\end{matrix}\right]dz \tag{4.4}$$

Now an appeal to the above results (4.3) and (4.4) gives,

$$\int_{-\infty}^{\infty}e^{-z^2}z^{2\rho}H_\mu(z)\ I^{0,n\ :(m_1,n_1);...;(m_r,n_r)}_{p_i,q_i:R:[p_{i'},q_i';R'];...;[p^{(r)}_{i(r)},q^{(r)}_{i(r)}:R^{(r)}]}\left[\begin{matrix}x_1\ z^{2\alpha}\\ \vdots\\ x_r\end{matrix}\middle|\begin{matrix}[(a_j;\alpha_j'...\alpha_j^{(r)})_{1,n}]\ \ [a_{ji};\alpha_{ji}'....\alpha_{ji}^{(r)})n+1,p_i]:[(c_j',\gamma_j')_{1,n_1}]\\ (b_j,;\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}]\ \ :[(d_j',\delta_j')_{1,m_1}]\end{matrix}\right.$$

$$\left.\begin{matrix}[(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}],[(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}})_{n_r+1,p_i^{(r)}}]\\ [(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}]\,[(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}})_{m_r+1,q_i^{(r)}}]\end{matrix}\right]dz$$

$$=\sum_{r=0}^{\infty}Q_r\int_{-\infty}^{\infty}e^{-z^2/2}H_r(z)\,H_\mu(z)\,dz$$

$$=(2\lambda)^{1/2}\,\mu!\,Q_4 \tag{4.5}$$

See orthogonal property of Hermite polynomials[4] [4, P. 289], which on further application of (3.1) gives,

$$Q_\mu=\frac{2^{\mu-2\rho-1/2}}{\mu!}I^{0,n\ :(m_1+1,n_1);...;(m_r,n_r)}_{p_i,q_i:R:[P_{i'}+1,\ q_i'+1:R'];...;[P_i^{(r)},q_i^{(r)}:R^{(r)}]}$$

$$\left[\begin{matrix} x_1 \, 2^{-2\alpha} \\ \vdots \\ x_r \end{matrix}\left| \begin{matrix} [(a_j;\alpha_j'...\alpha_j^{(r)})_{1,n}] \quad [a_{ji};\alpha_{ji}'....\alpha_{ji}^{(r)})_{n+1,p_i}]: \\ (b_{ji},;\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}] \end{matrix}\right.\right.$$

$$\left.\begin{matrix} [(c_j',\gamma_j')_{1,n_1}][(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}][(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,P_i^{(r)}}] \\ [(d_j',\delta_j')_{1,m_1}][(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}][(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{matrix}\right] dz \tag{4.6}$$

Thus by (4.3) solution of main problem is

$$\phi(z,t)=\sum_{r=0}^{\infty} e^{-(1-2r)kt-z^2t^2} H_{(r)}(z)\frac{z^{r-2\rho-1/2}}{r!} I^{0,n\,:(m_1+1,n_1);...;(m_r,n_r)}_{P_i,q_i:R:[P_{i'+1},q_i'+1':R'];...;[P_i^{(r)},\,q_i^{(r)}:\,R^{(r)}]}$$

$$\left[\begin{matrix} x_1 \, 2^{-2\alpha} \\ \vdots \\ x_r \end{matrix}\left| \begin{matrix} [(a_j;\alpha_j'...\alpha_j^{(r)})_{1,n}] \quad [a_{ji};\alpha_{ji}'....\alpha_{ji}^{(r)})_{n+1,p_i}],(-2\rho,2\alpha) \\ (b_{ji},;\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}] \qquad\qquad (\mu/2,\alpha) \end{matrix}\right.\right.$$

$$\left.\begin{matrix} [(c_j',\gamma_j')_{1,n_1}][(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}][(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,P_i^{(r)}}] \\ [(d_j',\delta_j')_{1,m_1}][(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}][(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{matrix}\right] dz$$

*5. Expansion formulae :*

Making an use (4.3), (4.4) and (4.6), we establish

$$z^{2\rho}e^{-z^2/2}\; I^{0,n\,:(m_1,n_1);...;(m_r,n_r)}_{P_i,q_i:R:[P_{i'},q_i':R'];...;[P_i^{(r)},q_i^{(r)}:R^{(r)}]}$$

$$\left[\begin{matrix} x_1 \, z^{2\alpha} \\ \vdots \\ x_n \end{matrix}\left| \begin{matrix} [(a_j;\alpha_j'...\alpha_j^{(r)})_{1,n}] \quad [(a_{ji};\alpha_{ji}'....\alpha_{ji}^{(r)})_{n+1,p_i}]; \\ (b_{ji},;\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}] \end{matrix}\right.\right.$$

$$\left.\begin{matrix} [(c_j',\gamma_j')_{1,n_1}][(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}][(c_{ji_{(r)}}^{(r)},\gamma_{ji_{(r)}}^{(r)})_{n_r+1,P_i^{(r)}}] \\ [(d_j',\delta_j')_{1,m_1}][(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}][(d_{ji_{(r)}}^{(r)},\delta_{ji_{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{matrix}\right] dz$$

$$= \sum_{r=0}^{\infty} \frac{2^{\gamma-2\rho-\frac{1}{2}}}{r!} I^{0,n\ :(m_1+1,n_1);...;(m_r,n_r)}_{[P_i,q_i:R]\ :[P_i',q_i':R'];.........;[P_i^{(r)},q_i^{(r)}:R^{(r)}]}$$

$$\left[\begin{matrix} x_1\ 2^{-\alpha} \\ \vdots \\ x_r \end{matrix}\middle| \begin{matrix} [(a_j;\alpha_j'...\alpha_j^{(r)})_{1,n}] \quad [(a_{ji};\alpha_{ji}'....\alpha_{ji}^{(r)})_{n+1,p_i}],(-2\rho,2\alpha);[(c_j',\gamma_j')_{1,n_1}], \\ (b_{ji},;\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}] \qquad (r/2-\rho,\alpha),[(d_j',\delta_j')_{1,m_1}], \end{matrix}\right.$$

$$\left.\begin{matrix} [(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}]\,[(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}]\,[(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{matrix}\right] dz \qquad (5.1)$$

Similarly making an appeal to (3.2) and orthogonal property of $P_\upsilon^\mu(z)$ [4, P, 278], we obtain the following expansion formula:

$$(1-z^2)^{\rho-1} I^{0,n\ :(m_1+1,n_1);...;(m_r,n_r)}_{[P_i,q_i:R]\ :[P_i',q_i':R'];........;[P_i^{(r)},q_i^{(r)}:R^{(r)}]}$$

$$\left[\begin{matrix} x_1\ (1-z^2)^{\alpha} \\ \vdots \\ x_r \end{matrix}\middle| \begin{matrix} [(a_j;\alpha_j'...\alpha_j^{(r)})_{1,n}] \quad [a_{ji};\alpha_{ji}'....\alpha_{ji}^{(r)})_{n+1,p_i}]; \\ (b_{ji},;\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}] \end{matrix}\right.$$

$$\left.\begin{matrix} [(c_j',\gamma_j')_{1,n_1}][(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}]\,[(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(d_j',\delta_j')_{1,m_1}][(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}]\,[(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{matrix}\right]$$

$$= \sum_{\upsilon=0}^{\infty} \frac{\pi(2\upsilon+1)(\upsilon-\mu)!2^{\mu-1}}{(\mu+\upsilon)!\Gamma\left(\frac{\mu+\upsilon}{2}+1\right)\Gamma\left(\frac{1-\mu-\upsilon}{2}\right)}$$

$$P_\upsilon^\mu(z)\ I^{0,n\ :(m_1+2,n_1+2);...;(m_r,n_r)}_{[P_i,q_i:R]\ :[P_i'+2,q_i'+2:R'];.........;[P_i^{(r)},q_i^{(r)}:R^{(r)}]}$$

$$\left[\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix}\middle| \begin{matrix} [(a_j,\alpha_j^{'}...\alpha_j^{(r)})_{1,n}] \quad (1-\rho-\mu/2,\alpha),(1+\rho+\mu/2,\alpha): \\ [(b_{ji},\beta_{ji}'...\beta_{ji}^{(r)})_{1,q_i}]:(-\rho-\upsilon/2,\alpha),\,(1-\rho+\upsilon/2,\alpha): \end{matrix}\right.$$

$$\left.\begin{matrix} [(c_j',\gamma_j')_{1,n_1}][(c_{ji}',\gamma_{ji}')_{n_1+1,P_i}];...;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}]\,[(c_{ji_{(r)}}^{(r)},\gamma_{ji_{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(d_j',\delta_j')_{1,m_1}][(d_{ji}',\delta_{ji}')_{m_1+1,q_i}];...;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}]\,[(d_{ji_{(r)}}^{(r)},\delta_{ji_{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{matrix}\right] \tag{5.2}$$

In the last we remark that I-function of several complex variables is most generalized function, therefore, specializing the number of variables and values of parameters, we can get results for different special functions of different variables.

## References


1. B.R. Bhonsle, Heat conduction and Hermite Polynomial, *Proc. Nat. Acad. Sci., India, 36*, 359 (1966).
2. R.C. Bohara and U.C. Jain, Integrals involving I- functions of several variables II, *Jhanabha, 12*, 97 (1982).
3. R.C. Singh Chandel and H.C. Yadav, Heat conduction and multiple Hypergeometric function of Srivastava and Daoust, Indian *J. Pure Appl. Math, 15 (4)*, 371 (1954).
4. A. Erdelgi *et al.*, Tables of integral transforms, vol. II, MC Graw-Hill (1954).
5. R.V. Charchil, operational Mathematics, Mc Graw Hill, New York (1958).
6. J.C. Jaeger, A continuation formula for 'Appell's function, F2, *J. Lond Math. Soc., 13* (1938).
7. Kempe' de feriet, Heat conduction and Hermite Polynomials, *Bull. Calcutta Math. Soc.*, The Golden Jubilee Commemoration Vol., 193 (1958-59).
8. N.N. Lebeder, Special functions and their applications, prentice Hall Inc. Engle wood, cliffs N.J. (1965).
9. B. Nager, P. Olsson and P. Weissgles, *Ark. Fys, 23*, 137 (1963).
10. P.M. Olsson, A Hypergeometric Function of two variables of importance in perturbation theory, I, II, *Ark Fys, 30*, 187, 29, 459 (1965).
11. E.D. Rainville, Special Functions, Macmillon company, New York (1965).
12. F. Singh, Expansion formulae for Kampe de Firiet and radial wave functions and heat conduction, *Def, Sci. Journal, 21*, 265 (1971).
13. H.M. Srivastava and M.C. Daoust, on Eulerion integrals associated with Kamp'e de Feriets function. *Publ. Inst. Math. (Beograd) Nouvelle Ser. 9 (23)* 499 (1969).
14. H.M. Srivastava and R. Pranda, Some Bilateral Generating functions for a class of generalized Hypergeometric Polynomials, *J. Reine Angew. Math, 283/284*, 265 (1976).
15. Sharma, C.K. and Ahmed : Vijnana Parishad Anusandhan Patrika (1997).