

Linearly Recursive Sequences of Fibonacci Numbers

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Abstract

The Fibonacci sequence is based on an additive relationship between any term and the three preceding terms. We shall make it a linear dependence and it will involve the preceding r terms. Here and throughout, q will note the general additive sequence.

$$q_{n+r} = a_1q_{n+r-1} + a_2q_{n+r-2} + \dots + a_rq_n \quad (n = 0, 1, 2, 3, \dots)$$

1 Introduction

The Fibonacci sequence has provoked a remarkable amount of research. It seems that there is no end to the results that may be derived from the basic definition⁷.

$$F_{n+2} = F_{n+1} + F_n \text{ and } F_0 = 0 \text{ and } F_1 = 1 \quad (1.1)$$

Which Leonardo of Pisa found lurking in the simple rabbit problem. An extension of the definition yields the following recurrence relation⁸:

$$P_{n+3} = P_{n+2} + P_{n+1} + P_n \text{ and } P_0 = 0, P_1 = 1 \text{ and } P_2 = 1 \quad (1.2)$$

Jaiswal D.V.² has extended Fibonacci recurrence relation to define the sequence $\{Q_n\}$, where,

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4}, \quad n \geq 4 \quad (1.3)$$

and Q_0, Q_1, Q_2, Q_3 are given arbitrary algebraic integers¹.

2 Linear Properties of Fibonacci Sequence:

The Fibonacci sequence is based on an additive relationship between any term and the three preceding terms. We shall make it a linear dependence and it will involve the preceding r terms. Here and throughout, q will note the general additive sequence^{3,6}.

$$q_{n+r} = a_1q_{n+r-1} + a_2q_{n+r-2} + \dots + a_rq_n \quad (n=0, 1, 2, 3, \dots) \quad (2.1)$$

It seems essential to the spirit of these sequences that they be integral. To ensure this, we must demand that the set

$$\{a_i\}_1^r$$

be integers. This set will be called the spectrum. But, returning to (2.1) and letting $n = 0$:

$$q_r = \sum_{k=1}^r a_k q_{r-k} \quad (2.2)$$

reveals that we must specify the first r terms of the sequence in order that the other may be obtained. The set of integers

$$\{q_i\}_0^{r-1}$$

so specified will be called the initial set of the initials.

It might be mentioned here that the Fibonacci sequence is obtained by letting $r = 4$ and taking the spectrum $\{1, 1, 2, 3\}$ and the initials $\{0, 1, 1, 2\}$.

We wish now to extend the definition (2.1) so that negative values for the index are allowed. Using the "back-up" approach, we obtain⁴

$$q_{r-1} = a_1 q_{r-2} + a_2 q_{r-3} + \dots + a_{r-1} q_0 + a_r q_{-1}$$

or

$$q_{-1} = \frac{1}{a_r} \left(q_{r-1} - \sum_{k=1}^{r-1} a_k q_{r-1-k} \right)$$

continuing, it can be seen that for any $n=0, 1, 2, \dots$

$$q_{-1} = \frac{1}{a_r} \left(q_{r-n} - \sum_{k=1}^{r-1} a_k q_{r-n-k} \right) \quad (2.3)$$

Clearly, in order to maintain an integral sequence for all values of the index, positive and negative, it is necessary to take $a_r = \pm 1$. In any case, we have that $a_r^2 = 1$.

3. Unary and Binary Sequences :

The number r of necessary initial

values classifies the sequence as unary, binary, tertiary, and so on. The analysis of the unary sequences is rather trivial. The spectrum is $\{a_1\}$ and the initial set $\{q_0\}$. But since $r = 1$, we must have $a_1 = \pm 1$, so that (2.1) comes down to :

$$q_{n+1} = \pm q_n$$

$$\text{Or, } q_{n+1} = (\pm)^{n+1} q_0$$

The analysis of the binary sequences is also trivial. The spectrum is $\{a_1, a_2\}$ and the initial set $\{q_0, q_1\}$. But since $r = 2$, we must have $a_1 = a_2 = \pm 1$, so that (2.1) comes down to :

$$q_{n+2} = \pm q_{n+2} \pm q_n$$

$$\text{Or, } q_{n+2} = (\pm)^{n+2} (q_1 \pm q_0)$$

In addition, it would seem altogether desirable to eliminate those sequences which can be "reduced" by dividing each term by a constant. That would leave only the primitive sequences for which if d divides q_k for each value of k , then $d = 1$.

4 Algebraic Generators :

One of the most common manifestations of additive recursive sequences is the power series expansion of certain functions. For example, a short calculation leads one to conclude that^{5,9} :

$$\frac{x}{1 - x - x^2 - x^3} = \sum_{k=0}^{\infty} p_k x^k$$

The actual derivation of this result stems directly from the definition of the Fibonacci sequence. In what follows, we will

use the same derivation in a generalized form.

What we want to discover is an expression for:

$$\sum_{k=0}^{\infty} q_k x^k = \phi(x) \tag{4.1}$$

Where , by (2.1)

$$q_{n+q} = \sum_{k=1}^r a_k q_{n+r-k} \tag{4.2}$$

Now, we multiply (2.1) by x^{n+r} , and sum over the index n, so that :

$$\sum_{n=0}^{\infty} q_{n+r} x^{n+r} = \sum_{k=1}^r a_k x^k \sum_{n=0}^{\infty} q_{n+r-k} x^{n+r-k} \tag{4.3}$$

But taking into (4.1) we may rearrange this expression, and:

$$\phi(x) \left(1 - \sum_{k=1}^r a_k x^k \right) = q_0 + \sum_{k=1}^{r-1} x^k \left(q_k \sum_{j=1}^k a_j q_{k-j} \right) \tag{4.4}$$

This singularly awkward expression can be made manageable by making the some what arbitrary definition of $a_0 = -1$. The introduction of a_0 greatly simplifies the formulation of the required function:

$$\phi(x) = \frac{\sum_{k=0}^{r-1} x^k \sum_{j=0}^k a_j q_{k-j}}{\sum_{k=0}^r a_k x^k} \tag{4.5}$$

We need hardly say that this is the required expression, which reduces to the familiar Fibonacci power series when $r=4$, $a_1=a_2=1$, $a_3=2$, $a_4=3$ and $q_0 = 0$, $q_1 = 1$ and $q_2 = 1, q_3=2$. But further investigation of (4.5)

leads to considerations which will be of crucial importance later. First, we remark that the denominator is a q-th degree polynomial¹ :

$$- a_0 + a_1x + a_2x^2 + \dots + a_r x^r, (a_0 = -1)$$

which will be called the spectral polynomial.

Then, with regard to the numerator, the following definition will be made :

$$g_{m,k} = - \sum_{r=0}^k a_j q_{m+k-j}, \text{ for } 0 \leq k \leq r \tag{4.6}$$

In other words, $g_{m,k}$ is a partial sum of terms, for example :

$$g_{m,0} = - a_0 q_m = q_m$$

$$g_{m,r-1} = a_r q_{m-1}$$

and, for convenience :

$$g_k = g_{0,k} = q_k - a_1 q_{k-1} - \dots - a_k q_0 \tag{4.7}$$

The introduction of (4.7) into (4.5) yields the remarkably concise:

$$\phi(x) = \sum_{k=0}^{\infty} q_k x^k = \frac{- \sum_{k=0}^{r-1} g_k x^k}{\sum_{k=0}^r a_k x^k} \tag{4.8}$$

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