Ultra Scientist Vol. 27(1)B, 33-38 (2015).

Approximation of Signal by Cesàro Operator Involving Lagrange Interpolating Polynomials on Jacobi Polynomials

S.P. YADAV and PREETI SINGH

Asstt. Prof, LNCT Group, Indore (India)
Spreetisingh1418@gmail.com

(Acceptance Date 9th April, 2015)

Abstract

In the present paper, we study approximation of signal cesàro operator involving Langrange interpolating polynomials. Here signal is a function which is use in form of the Jacobi Polynomials.

Key words: Approximation of Jacobi Polynomials, cesàro operator, Langrange interpolating polynomials.

2010 Mathematics Subject Classification: 41A10, 41A25

1 Introduction

Let C[-1, +1] be the set of all real valued continuous functions defined on the interval [-1,+1] and $P_n^{(\alpha,\beta)}(x)$ denotes the n^{th} Jacobi polynomial of order α, $\beta > -1$. The set $\{ (x) P_n^{(\alpha,\beta)} \}$ (n = 0,1,2….) is a complete orthogonal set of Jacobi polynomials having its zeros spread over [-1,+1].

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is defined as the solution of the homogenous differential equation of second order.

$$(1 - x^2)\, y + \{(\alpha + 1) - (\alpha + \beta + 2)\, x\}\, y + n(n+\alpha+\beta+1)\, y = 0 \qquad (1.1)$$

It is known (Szegö[4]) that the differential equation (1.1) has a polynomial solution

$$y = c\, P_n^{(\alpha,\beta)}(x) \qquad (1.2)$$

Which is not identically zero.

The Jacobi polynomials also satisfy the Rodrigues formula:

$$(1-x)^{\alpha}(1+x)^{\beta}\, P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\} \qquad (1.3)$$

α, β are arbitrary (see Szegö[4]). For $\alpha = \beta = 0$, the Jacobi polynomial reduces to the well known Legendre polynomial.

For a continuous function on the

interval [-1,+1], first we define the uniquely determined Lagrange interpolating polynomials of degree ≤ n-1 as follow

$$L_n^{(\alpha,\beta)}(f,x) =$$

$$\sum_{k=1}^{n} f\,(x_{kn}^{(\alpha,\beta)})\, l_k^{(\alpha,\beta)}(x) \qquad (1.4)$$

Where

$$l_k^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n'^{(\alpha,\beta)}(x_{kn}^{(\alpha,\beta)})(x - x_{kn}^{(\alpha,\beta)})} \qquad (1.5)$$

And $x_{kn}^{(\alpha,\beta)} = x_k$ (k = 1,2….,n) are roots of $P_n^{(\alpha,\beta)}$.

$$-1 < x_{n,n}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots.. < x_{k,n}^{(\alpha,\beta)} \ldots < x_{2n}^{(\alpha,\beta)} < x_{1n}^{(\alpha,\beta)} < 1 \qquad (1.6)$$

It is well known that

$$\sum_{k=1}^{n} l_k^{(\alpha,\beta)}(x) = 1 \text{(see Szegö)}^{4}, (14.1.5)) \qquad (1.7)$$

(n = 1,2,…)

We have studied a polynomial (say) "Cesàro Lagrange interpolating Polynomial", which has better convergence property in [-1,+1]. We define it in the following way[2,5,6]:

$$S_n^1(f,L,x) = \frac{1}{n} \sum_{v=1}^{n} L_v^{(\alpha,\beta)}\, f\,(x) \qquad (1.8)$$

Now, by $L_n^{(\alpha,\beta)}(1, x) = 1$, we have

$$|\,f(x) - S_n^1(f,L,x)| = |\,f(x) - \frac{1}{n} \sum_{v=1}^{n} L_v^{(\alpha,\beta)}\, f\,(x)|$$

$$= \left| f(x)\, \frac{1}{n} \sum_{v=1}^{n} L_v^{(\alpha,\beta)}(1, x) - \frac{1}{n} \sum_{v=1}^{n} L_v^{(\alpha,\beta)}\, f(x) \right|$$

$$= \frac{1}{n} \left| \sum_{v=1}^{n} \{ f(x) - L_v^{(\alpha,\beta)}(f,x) \} \right|$$

$$\leq \frac{1}{n} \sum_{v=1}^{n} |\, f(x) - L_v^{(\alpha,\beta)}(f,x) \}|$$

$$= \frac{1}{n} \sum_{v=1}^{n} \left| f(x) - \sum_{k=1}^{v} f\,(x_{kv}^{(\alpha,\beta)})\, l_k^{(\alpha,\beta)}(x) \right|$$

$$= \frac{1}{n} \sum_{v=1}^{n} \left| f(x) \sum_{k=1}^{v} l_k^{(\alpha,\beta)}(x) - \sum_{k=1}^{v} f\,(x_{kv}^{(\alpha,\beta)})\, l_k^{(\alpha,\beta)}(x) \right|$$

$$= \frac{1}{n} \sum_{v=1}^{n} \left| \sum_{k=1}^{v} (f(x) - f\,(x_{kv}^{(\alpha,\beta)}))\, l_k^{(\alpha,\beta)}(x) \right|$$

$$= \frac{1}{n} \sum_{v=1}^{n} \sum_{k=1}^{v} \left| f(x) - f\,(x_{kv}^{(\alpha,\beta)}) \right| .\, l_k^{(\alpha,\beta)}(x) \qquad (1.9)$$

2. Main Results

Concerning the convergence of Lagrange interpolation Vertesi, P.O.H.[3] has given the following theorem.

Theorem: If x is an arbitrary point from (-1,1) then there exists an $f(x) \in C[-1,1]$ a sequence $\{\omega_m\}$ and $0 < n_1 < n_2 <\ldots$. Such that

$$L_n^{(\alpha,\beta)}(f,x^*) - f(x^*) > \log n\omega_n \left(\frac{1}{n}\right) \qquad (2.1)$$

($n=n_1, n_2, n_3\ldots.., x^* \neq \pm 1;\ \alpha, \beta > -1;\ x^* \in (-1,1)$)

We consider a subspace lip δ (0 < δ < 1) of C [-1, +1) such that $f \in$ C [-1,+1] for all $x, y \in$ [-1,+1], there exists a constant M, so that

$$|\, f(x) - f(y) \,| \leq M\, |\, x - y\,|^{\delta}, \ (0 < \delta < 1) \qquad (2.2)$$

It is natural to think beyond the continuity of the function. In this regard, for $f \in$ lip δ (0

$< \delta < 1$), we not only find the convergence of the operator S_n^1 $(f, L,, x)$ but we also deduce the rate of convergence which we call the degree of convergence of S_n^1. Moreover it is traced that the function f is represented by the interpolatary polynomial S_n^1. Precisely, we prove the following:

Theorem 1: Let $f \in \text{lip}\ \delta$ $(0 < \delta < 1)$ Then

$$\| f(x) - S_n^1 (f, L, x) |_{[-1,1]} = O(1) \quad (2.3)$$

for $-1 < \alpha \leq -½$, and β arbitrary, $S_n^1 (f, L, x)$ is defined by (1.8)

To prove the theorem we need the asymptotic orders of zeros of Jacobi polynomial and certain known results about Jacobi polynomials, which are as follows.

Lemma 1: Szegö[4] page 169 (7.32.5)

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-½}\, O(n^{-½}) \ ; & c/n \leq \theta \leq \pi/2 \\ O(n^{\alpha}) & : 0 \leq \theta \leq c/n \end{cases} \quad (2.4)$$

Some known results from Szegö[4]

$\theta_k^{(\alpha,\beta)} = \theta_k = \frac{1}{n}[k\pi + o(1)]$ $(k = 1, 2, \ldots\ldots, n\ ; n = 1, 2, \ldots.)$ (2.5)

$|P_n'^{(\alpha,\beta)}(\cos\theta_k)| \sim k^{-\alpha-3/2}\, n^{\alpha+2}$ $(0 < \theta_k \leq \pi/2;\ n = 1, 2, \ldots)$ (2.6)

Let $x = \cos\theta$, $x_k = \cos\theta_k$ $(k = 1, 2, \ldots.n)$
$x_0 = 1$, and $x_{n+1} = -1$
Then

$\frac{c_1}{n} \leq \theta_{k+1} - \theta_k \leq \frac{c_2}{n}$ $(k = 0, 1, \ldots n)$ (see Vertesi[3]) (2.7)

with $0 < c_1 = c_1(\alpha,\beta)$ and $c_2 = c_2(\alpha,\beta)$

and $| x - x_k | \sim n^{-2} | j^2 - k^2 |$ if $x \in [x_{j+1}, x_j]$

$(k = 0, 1, \ldots.\ n+1\ ;\ k \neq j, j+1$ (see Natanson's G.I.[1] Lemma 1 and 2)

Further

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x) \quad (2.8)$$

$$\lambda_{k,n}^{(\alpha,\beta)} = \lambda_k \sim (1 - x_k^2)\, [P_n'^{(\alpha,\beta)}(x_k)]^{-2} \quad (2.9)$$

$$\lambda_k \sim \theta_k^{2\alpha+1} n^{-1} \sim k^{2\alpha+1} n^{-2\alpha-2} \ (0 < \theta_k \leq \pi - \varepsilon) \quad (2.10)$$

where λ_k are the christoffel number (Szegö[4], (15.3.14)).

If $x_{jn} = x_j$ be the nearest root of x, then we write

$$i = | k - j | \text{ for } k \neq j \text{ and } i = 1 \text{ for } k = j \quad (2.11)$$

We often use

$$| \theta - \theta_k | \sim \frac{i}{n},\ (k \neq j) \quad (2.12)$$

Proof of the theorem 1 :

For $f \in \text{lip}\ \delta$ $(0 < \delta < 1)$, we have

$$| f(x) - f(x_k) | \leq M\, | x - x_k |^{\delta}$$

Then by (1.9), we get

$$\| f(x) - S_n^1 (f, L, x) \|_{[-1,1]} \leq M\, n^{-1} \sum_{V=1}^{n} \left[\sum_{k=1}^{v} | x - x_k|^{\delta}\, l_k^{(\alpha,\beta)}(x) \right]$$

To make the analysis crystal clear, we estimate the interior sum " $\sum_{k=1}^{v}$ " as given below for α replaced by α + 1. s

Thus,

$$\sum_{k=1}^{v} | x - x_k |^{\delta} l_k^{(\alpha,\beta)} (x) = O\left\{ \sum_{k=1}^{v} | x - x_k |^{\delta} \frac{P_v^{(\alpha+1,\beta)}(x)}{\{P_v'^{(\alpha+1,\beta)}(x_k)\}(x - x_k)} \right\}$$

where $x_k = x_{kv}^{(\alpha+1,\beta)}$ is the k^{th} zero of the v^{th} polynomial $P_v^{(\alpha+1,\beta)}(x)$.

$$= \sum_{k=j} + \sum_{|x-x_k|<\epsilon\ (\alpha+1,\beta)} + \sum_{x-x_k>\epsilon\ (\alpha+1,\beta)} + \sum_{x_k-x>\epsilon\ (\alpha+1,\beta)}$$

$$I + II + III + IV$$

Now, we consider I

For $k = j$, we have

$$\frac{P_v^{(\alpha+1,\beta)}(x) - P_v^{(\alpha+1,\beta)}(x_j)}{(x - x_j)} = P_v'^{(\alpha+1,\beta)}(x_j)$$

Then,

$$P_v^{(\alpha+1,\beta)}(x) = (x - x_j)\, P_v'^{(\alpha+1,\beta)}(x_j)$$

$$\{\text{since } P_v^{(\alpha+1,\beta)}(x_j) = 0\}$$

So,

$$I = O\,\{|x - x_j|^{\delta}\,.1\}$$

$$= O\left\{\left(\frac{1}{v}\right)\right\}$$

$$= O(v^{-\delta}) \qquad (2.13)$$

Now we discuss II.

Since $|x - x_k| < \varepsilon\,(\alpha+1,\beta) \Rightarrow k = n$ and $|x - x_k| \sim \frac{k}{v}$

Let $x - \cos\theta$, $x_k - \cos\theta_k$ and here we will conclude for $\theta_k < \theta$ and $\theta < \theta_k$ respectively, so

$$II = \underset{\theta_k<\theta}{II_1} + \underset{\theta<\theta_k}{II_2}$$

First we discuss II_1

Here $\theta_k < \theta$

Case (a): $\frac{c_1}{2n} \leq \theta \leq \frac{\pi}{2}$, $2\theta_k > \theta$ but $\theta_k < \theta$, so $\theta \sim \theta_k$

Then

$$II_1 = O\left\{ \sum_{k=1}^{v} | x - x_k |^{\delta} \frac{P_v^{(\alpha+1,\beta)}(x)}{P_v'^{(\alpha+1,\beta)}(x_k)(x - x_k)} \right\}$$

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta-1} \frac{\theta^{-\alpha-3/2} v^{-½}}{k^{-\alpha-5/2}\, v^{\alpha+3}} \right\} \text{using (2.4) and (2.6)}$$

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta-1} \frac{k^{-\alpha-3/2} v^{-½}}{v^{-\alpha-3/2} k^{-\alpha-5/2}\, v^{\alpha+3}} \right\} \left(\text{since } \theta \sim \theta_k = \frac{k}{v}\right.$$

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta-1} k.v^{-2} \right\}$$

$$= O\left\{ \sum_{k=1}^{v} k^{\delta} v^{-\delta-1} \right\}$$

$$= O(1) \qquad (2.14)$$

Case (b) : $\frac{c_1}{2n} \le \theta \le \frac{\pi}{2}$, $2\theta_k < \theta$, so $\theta_k = O(\theta)$

Then

$$II_1 = O\{\sum_{k=1}^{v} \; |x - xk|\delta \; \frac{P_v^{(\alpha+1,\beta)}(x)}{P_v^{\prime(\alpha+1,\beta)}(x_k)(x-x_k)}\}$$

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta-1} \frac{\theta^{-\alpha-3/2} v^{-½}}{k^{-\alpha-5/2}\, v^{\alpha+3}} \right\}$$

$$= O(1) \quad \text{(as estimated earlier)} \qquad (2.15)$$

Now, we calculate II_2

Here $\theta < \theta_k$

Case (a): $0 \le \theta \le \frac{c_1}{2n}$ Now by (2.7), $2\theta < \theta_k$

So, $x - x_k = \cos\theta - \cos\theta_k \le \theta_k^2$

i.e. $|x - x_k| \le \frac{k2}{v2}$ and $x - x_k > 0$ (2.16)

So

$$II_2 = O\left\{ \sum_{k=1}^{v} |x - x_k|^{\delta} \frac{P_v^{(\alpha+1,\beta)}(x)}{P_v^{\prime(\alpha+1,\beta)}(x_k)(x-x_k)} \right\}$$

$$= O\left\{ \sum_{k=1}^{v} |x - x_k|^{\delta-1} \frac{v^{\alpha+1}}{k^{-\alpha-5/2}\, v^{\alpha+3}} \right\}$$

Using (2.4), (2.6) and (2.7)

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k2}{v2} \right|^{\delta-1} v^{-2} k^{\alpha} \right\}$$

$$= O\left\{ v^{-2\delta} \sum_{k=1}^{v} k^{2\delta+\alpha+1/2} \right\}$$

$$= O\left\{ v^{-2\delta} v^{2\delta+\alpha+3D\ 2} \right\}$$

$$= O\left\{ v^{\alpha+3/2} \right\} \qquad (2.17)$$

Case (b) : $\frac{c_1}{2n} \le \theta \le \frac{\pi}{2}$, $2\theta \ge \theta_k$, but $\theta < \theta_k = \theta \sim \theta_k$

$$II_2 = O\left\{ \sum_{k=1}^{v} |x - x_k|^{\delta-1} \left(\frac{\theta^{-\alpha-3/2} v^{-½}}{k^{-\alpha-5/2}\, v^{\alpha+3}} \right) \right\}$$

using (2.4) and (2.6)

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta-1} k^{-\alpha-3/2+\alpha+5/2} v^{\alpha+3/2-1/2-\alpha-3} \right\}$$

$\{\text{since } \theta \sim \theta_k = \frac{k}{v}\}$

$$= O\{ \sum_{k=1}^{v} v^{-\delta-1} k^{\delta} \}$$

$$= O(1) \qquad (2.18)$$

Case (c) : $\frac{c_1}{2n} \le \theta \le \frac{\pi}{2}$ and $2\theta \le \theta_k$. As in case (a), we get

$$II_2 = O\{v^{\alpha+3/2}\} \qquad (2.19)$$

On combining (2.14) to (2.19), we get

$$II = O\{v^{\alpha+3/2}\} \qquad (2.20)$$

Now we take III

$$III = O\left\{ \sum_{\substack{k=1 \\ x-x_k > \epsilon}}^{v} |x - x_k|^{\delta} \frac{P_v^{(\alpha+1,\beta)}(x)}{P_v^{\prime(\alpha+1,\beta)}(x_k)(x-x_k)} \right\}$$

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta} \frac{\theta^{-\alpha-3/2} v^{-½}}{k^{-\alpha-5/2}\, v^{\alpha+3}.\varepsilon} \right\}$$

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta} k^{-\alpha-3/2+\alpha+5/2} v^{\alpha+3/2-1/2-\alpha-3} \right\}$$

$$= O\left\{ \sum_{k=1}^{v} \left| \frac{k}{v} \right|^{\delta} k v^{-2} \right\}$$

$$= O(1) \qquad (2.21)$$

At last we estimate *IV*

$$IV = O\left\{\sum_{x_k - x > \epsilon} |x-x_k|^{\delta}\ l_k^{(\alpha+1,\beta)}\right\}$$

Since $x_k - x > \varepsilon > 0$

So $\frac{1}{x-x_k} < \frac{1}{\varepsilon}$

Thus it can also be estimated as in case III, that is

$$IV = O(1) \qquad (2.22)$$

On combining the results obtained in (2.13), (2.20), (2.21) and (2.22), we get

$$\sum_{k=1}^{v} |x-x_k|^{\delta}\ l_k^{(\alpha+1,\beta)}(x) = O(v^{-\delta}) + O(v^{\alpha+3/2}) + O(1)$$

Now replacing α+1 by α, we get

$$\sum_{k=1}^{v} |x-x_k|^{\delta}\ l_k^{(\alpha+1,\beta)}(x) = O(v^{-\delta}) + O(v^{\alpha+1/2}) + O(1)$$

Thus, we have

$$\| f(x) - S_n^1(f, L, x) \|_{-1,1} = O(v^{-\delta}) + O(v^{\alpha+½}) + O(1)$$

For $0 < \delta < 1$, under supremum norm

$$\| f(x) - S_n^1(f, L, x) \|_{-1,1} = O(1)$$

For all $-1 \le \alpha \le -½$ and β arbitrary.

This completes the proof of theorem 1.

References

1. Natanson's G.I., Two sided estimate for Lebesgue function of the Lagrange interpolation with Jacobi nodes.I_{20} Vyss Ucebn, Zaved, **Mathematika.** *11*, 67-74 (1967). (Russion).
2. Resnikoff, Howard L. and Wells Jr., Raymond O. Wavelet Analysis, The Scalable Structure of Information. Spinger-verlog New York, Inc. (1998).
3. Vertesi, P.O. Notes on the Hermita-Fezer interpolation based on the Jacobi Abscissas. **Actad. Mathematica Acad. Scientiarum Hungarica Tomus.** 24, pp. 233-239 (1973).
4. SzegÖ,G. Orthogonal Polynomials. Amer. Math. Soc., Collo. Pub., Vol. XXIII 3rd ed., New York (1967).
5. Tian, J. and Wells Jr., R. O., Vanishing Moment and Biorthogonal Coifman Wavelet System.
6. Yadav, Sarjoo Prasad., On a Banach Space Approximable by Jacobi Polynomials. Acta Math. Hungar. *98*(1-2), 21-30 (2003).