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Approximation of Signal in Lebesgue Function by a System of Wavelets

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Abstract

In the present paper, we study approximation of signal in Lebesgue function by a system of wavelet. Here wavelet i.e. in terms of Coifman wavelet system and signal is a function which is use in form of the Legendre polynomials. We have used Legendre polynomials, thus general case jacobi polynomials.

Key words: Approximation of Legendre polynomials, Coifman wavelet system.

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1-Introduction

We write X to denote both of the spaces C and $\boldsymbol{L^P}$. C is the space of all continuous functions on [-1, 1] and $\boldsymbol{L^P}$ $(1 \leq p < \infty)$ are space of p-power Lebesgue integrable functions on [-1, 1]. The expansion of function $f(x) \in X$ at $x = \cos\theta$ in term of Legendre polynomials is given by

$$\boldsymbol{f(cos\theta)} \sim \sum_{\boldsymbol{n=0}}^{\infty} \boldsymbol{a_n P_n} \ (\boldsymbol{cos\theta}) \qquad (1.1)$$

$$≡ \sum u_n (\cos\theta), \qquad \text{(say)}$$

Where

$$a_n = (n + \tfrac{1}{2}) \int_0^{\pi} f(cos\omega)\, sin\omega P_n(cos\omega) d\omega \qquad (1.2)$$

$P_n(\cos\theta)$ is n^{th} Legendre Polynomial and the integral in (1.2) is presumed to exist.
We write

$$f(\omega) = \{f(cos\omega) - A\} \qquad (1.3)$$

$$\sigma_n(f, \theta, X) \equiv \sigma_n(f) = \sum_{v=0}^{n} a_{nv} s_v(cos\theta) \qquad (1.4)$$

And

$$s_v(cos\theta) = \sum_{k=0}^{n} a_k\, p_k(cos\theta) \qquad (1.5)$$

is the v^{th} partial sum of the series (1.1). $a_{n,v}$ is an element of n^{th} row and v^{th} column of the

lower-triangular infinite matrix.
eg.

$$a_{n,k} \geq 0, (n,k = 0,1,2,\ldots), \sum_{k=0}^{n} a_{n,k} = 1 \quad (1.6)$$

$$a_{n,k} \geq a_{n,k+1}, (k=0,1,\ldots\ldots, n-1; n=0,1,\ldots \quad (1.7)$$

Theorem A. Let $(a_{n,k})$ satisfy (1.6) and (1.7) and $\omega(t)$ and $H(t)$ be such that

$$\int_u^{\pi} \frac{t^{-1}\omega(t)}{\cos\theta - \cos t}\, dt = 0\{H(u)\}, u \to 0^+ \quad (1.8)$$

Where

$$H(u) \geq 0 \quad (1.9)$$

And

$$\int_0^t H(u)du = O\{t\, H(t)\}, \quad (t \to 0^+) \quad (1.10)$$

Then

$$\|\sigma_n(f) - f\|_X = O\{a_{n,0} H(a_{n,0})\} \quad (1.11)$$

In case, we choose a matrix such that $a_{n,0} \to 0$ as $n \to \infty$, then

$$\|\sigma_n(f) - f\|_X = O(1) \ \text{as}\ n \to \infty \quad (1.12)$$

Where $\sigma_n(f)$ is the matrix transform of the series (1.1) as given in (1.4).
Thus

$$\sigma_n(f,\theta,X) \equiv \sigma_n(f) = \sum_{v=0}^{n} a_{nv} \sum_{k=0}^{v} a_k P_k(\cos\theta)$$

$$= \sum_{k=0}^{v} a_k P_k(\cos\theta) \sum_{v=k}^{n} a_{n,k}$$

$$= C_0 P_0(\cos\theta) + C_1 P_1(\cos\theta) + \ldots + C_r P_r(\cos\theta) + \ldots C_n P_n(\cos\theta)$$

$$= \sum_{k=0}^{v} C_r P_r(\cos\theta) \quad (1.13)$$

Where

$$C_r = (a_r \textstyle\sum_{v=r}^{n} a_{n,v}) \quad (1.14)$$

Thus $\sigma_n(f)$, define as above, is a linear combination of Legendre Polynomials of atmost degree n and therefore it is a polynomial of degree n.

We know that the Legendre Polynomial $P_n(x)$ is define as:

$$P_0(x) = 1 \quad (1.15)$$

$$P_1(x) = x$$

and recurrence relation any three consecutive Legendre Polynomials is given by

$$(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$$

(see szegæ□[2], page 71, eqn.(4.5.1) for $\alpha=0=\beta$)

Thus the Polynomial $\sigma_n(f)$ given in (1.13) is a Polynomial in $x = \cos\theta$ and its degree is n. It is n-times differentiable and its n^{th} derivative is constant. Its N^{th} $(N > n)$ derivative which will be zero, exhibit that it is continuous. By $C_o^N(R)$ we mean the space of all function $f(x)$ which are n times differentiable with N^{th} derivative continuous and tending to zero as $x \to \infty$ on R. In all we conclude the following proposition.

Proposition: The polynomials $\sigma_n(f,\theta,X)$ given in (1.12) is N $(N>0)$ times differentiable and N^{th} derivative is continuous on R.
This show that

$$\sigma_n(f)\, \chi_{[-1,1]}(x) \in C_o^N(R) \quad (1.16)$$

Where $\chi_{[-1,1]}(x)$ is characteristics function on [-1.1].

We apply the following theorem of Tian and Wells[3] Jr. (see Resnikoff and Wells[3], Jr., page 206) to get our approximation results.

Theorem T_1:

For an orthogonal Coifman wavelet system of degree N with the scaling function $\varphi(x)$ and scaling vector α, assume α has finite length. If $f(x) \in C_0^N$ (R), define for $j \in N$

$$S^j(f)(x) = 2^{-j/2} \sum_{k \epsilon Z} f\left(\frac{k}{2^j}\right) \varphi_{jk}(x) \quad (1.17)$$

Then

$$\| f(x) - S^j(f)(x) \|_{L^2} \leq C\, 2^{-jN} \quad (1.18)$$

Where C depends only on *f(x)* and the scaling vector α. $S^j(f)(x)$ is called **wavelet Sampling Approximation** of the function *f(x)* at the level *j*.

Theorem T_2:

Under the same hypothesis as in theorem T_1

$$\| f(x) - P^j(f)(x) \|_{L^2} \leq C\, 2^{-jN} \quad (1.19)$$

Where C depends only on *f(x)* and the scaling vector α, and

$$P^j(f) = \sum_{k \epsilon Z} \left(\int_R \ (f(x) \varphi_{jk}\, x\, dx \right) \varphi_{jk}(x) \quad (1.20)$$

is called **Wavelet Orthogonal Projection**.

2. Main Result

Theorem 1 :

For an orthogonal Coifman wavelet system of degree N with the scaling function *φ(x)* and the scaling vector α, assume α has finite length. If $f \ \epsilon \ L^P$ $(1 \leq p \leq \infty)$, for $p = \infty$ it is assumed that f is continuous. Define for $j \ \epsilon \ N$, the **wavelet Sampling Approximation**

$$S^j(\sigma_n \chi_{[-1,1]})(x) = 2^{-j/2} \sum_{k \epsilon Z} \left[\sigma_n\left(\frac{k}{2^j}\right) \chi_{[-1,1]}\left(\frac{k}{2^j}\right) \right] \varphi_{jk}(x) \quad (2.1)$$

Then, there exist $n_0 \ \epsilon$ N such that

$$\| f(x) - S^j (\sigma_n \chi_{[-1,1]})(x) \|_X \leq C\, 2^{-jN} \quad (2.2)$$

Where C depends only on *f(x)* and the scaling vector α. $\varphi(x)$ is scaling function while the corresponding orthogonal wavelet basis is $\{\varphi_{jk}(x)\}$.

$S^j(\sigma_n \chi_{[-1,1]})(x)$ is called the **Wavelet sampling approximation** of the function *f(x)* at the level *j*. Here we denote $(fg)(x) = f(x)\, g(x)$. Again we define the **Wavelet Orthogonal Projection** with respect to function σ_n as :

$$P^j\left(\sigma_n(x) \chi_{[-1,1]}(x)\right) = \sum_{k \epsilon Z} \left(\int_{-1}^{1} \sigma_n(t) \varphi_{jk}(t) dt \right) \varphi_{jk}(x) \quad (2.3)$$

Our next theorem on Approximation by Coifman wavelet system is given below.

Theorem 2:

Under the same hypothesis as in theorem 1 if $f \epsilon L^P$ $(1 \leq p \leq \infty)$, then there exist an integer $n_0 \ \epsilon \ N$ such that

$$\| f(x) - P^j (\sigma_n \chi_{[-1,1]})(x) \|_X \leq C\, 2^{-jN} \quad (2.4)$$

where the constant, C depends only on f and scaling vector α, and P^j is as given in (2.3).

Theorem 2: By (1.12), we have

$$\| \sigma_n(f) - f \|_x = 0(1) \quad \text{as } n \to \infty$$

i.e. for every $\varepsilon > 0$, there exists an integer n_0 such as that $n \geq n_1$, then

$$\| \sigma_n(f) - f \|_X < \varepsilon \qquad (2.5)$$

Thus, for every positive integer j and N, setting $\varepsilon = 2^{-jN}$, we have

$$\| \sigma_n(f) - f \|_X < C_1 2^{-jN} \qquad (2.6)$$

Now, for $n_0 \geq n_1$,

$$\| f(x) - S^j (\sigma_{n_0} \chi_{[-1,1]})(x) \|_X \leq \| f(x) - \sigma_n(f) \|_X$$

$$+ \| \sigma_n(f) - S^j (\sigma_{n_0} \chi_{[-1,1]})(x) \|_X$$

$$\leq \varepsilon + c \| \sigma_{n_0}(f) - S^j (\sigma_{n_0} \chi_{[-1,1]})(x) \|_{L^2}$$

$$\leq C_2 2^{-jN} + C_3 2^{-jN}$$ (by (2.6) and theorem T_1 for $n_0 \geq n_1$)

$$\geq C\, 2^{-jN}$$

As it has been shown that $\sigma_n(f) \in C_0^N(R)$. Therefore X and L^2 norms are defined for σ_n and S^j.

This completes the proof of theorem 1.

Proof of theorem 2 : In view of the proof of theorem T_2, it is clear that the proof of theorem 2 follows on the lines of theorem 1.

Thus the proof of the theorem 2 is complete.

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