Ultra Scientist Vol. 27(2)B, 85-94 (2015).

Groupoid Matrix Semirings and Their Properties

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(Acceptance Date 24 June, 2015)

Abstract

In this paper for the first time groupoid matrix semirings are introduced and their properties are analysed. A matrix groupoid or a groupoid matrix is a collection of p × q matrices with entries from a groupoid (G, *) where groupoid matrix inherits the operation of G. Here chain lattices are used as semirings and groupoid matrix semirings are non associative semirings of finite order, as both the groupoids G and the chain lattices are of finite order.

Key words: Groupoid semirings, groupoid matrix, chain lattice, groupoid, matrix groupoid, semiring, groupoid matrix semiring.

1. Introduction

This paper is organized into three sections. Section one is introductory in nature. Section two defines groupoid matrix semirings and the special properties associated with them are derived. The final section give conclusions based on this study. Here we just recall the definition of matrix groupoids and give some examples of them.

*Definition 1.1: Let (G, *) be any groupoid. M = {Collection of all p × q matrices with entries from G;* $\times_n$*} is defined as the matrix groupoid. (*$\times_n$ *operation acts natural product on matrices with induced operation ** [4]*).*

We will illustrate this by an example.

Example 1.1: Let {G, *} = {Z_5, *, (4, 1)} be a groupoid.

$$M=\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| a_i \in Z_5, 1\le i\le 3, \times_n\right\}$$

is a matrix groupoid. Here if

$$x=\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } y=\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \in M.$$

$$x \times_n y = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \times_n \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3*0 \\ 2*1 \\ 1*3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}.$$

Throughout this paper we use only groupoids built using Z_n.

2. Groupoid matrix semirings and their properties :

In this section the notion of groupoid matrices (or matrix groupoid using Z_m is defined using the groupoids, $G=\{Z_m, *, (t, s), t, s \in Z_m\}$. The semirings taken in this section are only finite chain lattices; $C_n = 0 < a_1 < a_2 < \ldots < a_{n-2} < 1$ (n a positive finite integer). Let $M_{p \times q}$ = {Collection of all p × q matrices with entries from Z_m; (t, s), *; t, s $\in Z_m$, $\times_n$} be the groupoid matrices of order p× q. $C_n M_{p\times q}$ is the groupoid matrix semiring defined as in case of usual groupoid semirings introduced in [1]. First a few examples are given for the better understanding of this concept.

Example 2.1: Let M = {Collection of all 1 × 3 row matrices with entries from Z_9, (3, 6), ×} be the groupoid row matrix (row matrix groupoid).

Let $C_5 = 0 < a_1 < a_2 < a_3 < 1$ be the chain lattice of order 5. C_5M is the groupoid row matrix semiring of the groupoid matrix M over the semiring C_5.

Let

$$x = a_1(2, 0, 1) + a_2(0, 0, 7) \text{ and } y = (7, 8, 4) + a_3 (1, 2, 3) \in C_5M.$$

$$x + y = a_1 (2, 0, 1) + a_2 (0, 0, 7) + (7, 8, 4) + a_3 (1, 2, 3) \in C_5M.$$

$$\begin{aligned} x \times y &= (a_1 (2, 0, 1) + a_2 (0, 0, 7)) * \{(7, 8, 4) + a_3 (1, 2, 3)\} \\ &= a_1 \cap 1 [(2, 0, 1) * (7, 8, 4)] + a_1 \cap a_3 ((2, 0, 1) * (1, 2, 3)) + \\ &\quad a_2 \cap 1 ((0, 0, 7) * (7, 8, 4)) + a_2 \cap a_3 ((0, 0, 7) * (1, 2, 3)) \\ &= a_1 (2 \times 3 + 7 \times 6, 0 \times 3 + 8 \times 6, 1 \times 3 + 4 \times 6) + a_1 (2 \times 3 + 1 \times 6, 0 \times 3 \\ &\quad + 2 \times 6, 1 \times 3 + 3 \times 6) + a_2 (0 \times 3 + 7 \times 6, 0 \times 3 + 8 \times 6, 7 \times 3 + 4 \times 6) \\ &\quad + a_2 (0 \times 3 + 1 \times 6, 0 \times 3 + 2 \times 6, 7 \times 3 + 3 \times 6) \\ &= a_1 (3, 3, 0) + a_1 (3, 3, 3) + a_2 (6, 3, 0) + a_2 (6, 3, 3) \in C_5M. \end{aligned}$$

This is the way × is defined on C_5M.

Let

$$\begin{aligned} x &= a_1 (3, 3, 5) \in C_5M; \\ x + x &= (a_1 \cup a_1) (3, 3, 5) \\ &= a_1 (3, 3, 5) \\ &= x. \\ x \times x &= a_1 (3, 3, 5) \times a_1 (3, 3, 5) \\ &= a_1 \cap a_1 [(3, 3, 5) * (3, 3, 5)] \\ &= a_1 (3\times3+3\times6, 3\times3+3\times6, 5\times3+5\times6) \\ &= a_1(0, 0, 0). \end{aligned}$$

Thus x is a zero matrix. Infact for every $x = a_i (x_1, x_2, x_3) \in C_5M$; $a_i \in C_5$; $(x_1, x_2, x_3) \in M$. x × x = (0). Thus in C_5M every element α such that |supp x| = 1 is such that x × x = (0).

Example 2.2: Let

$$P=\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in Z_{12}, 1 \le i \le 4, (8, 4), \times_n\right\}$$

be the 4 × 1 column matrix groupoid; $C_2=\{0,1\} = 0 < 1$ be the two element chain lattice. C_2P be the groupoid matrix semiring.

Let

$$x = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 0 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 7 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 9 \end{bmatrix} \in C_2.$$

$$x \times_n y = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 7 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 5 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 3 \\ 5 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 5 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3*7 \\ 2*2 \\ 5*0 \\ 0*1 \end{bmatrix} + \begin{bmatrix} 3*3 \\ 2*5 \\ 5*1 \\ 0*3 \end{bmatrix} + \begin{bmatrix} 3*1 \\ 2*0 \\ 5*0 \\ 0*9 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 8 + 7\times 4 \\ 2\times 8 + 2\times 4 \\ 5\times 8 + 0\times 4 \\ 0\times 8 + 1\times 4 \end{bmatrix} + \begin{bmatrix} 3\times 8 + 3\times 4 \\ 2\times 8 + 5\times 4 \\ 5\times 8 + 1\times 4 \\ 0\times 8 + 3\times 4 \end{bmatrix}$$

$$+ \begin{bmatrix} 3\times 8 + 1\times 4 \\ 2\times 8 + 0\times 4 \\ 5\times 8 + 0\times 4 \\ 0\times 8 + 9\times 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 4 \\ 0 \end{bmatrix} \in C_2P.$$

Let

$$x = \begin{bmatrix} 7 \\ 3 \\ 1 \\ 2 \end{bmatrix} \in C_2P.$$

$$x \times_n x = \begin{bmatrix} 7 \\ 3 \\ 1 \\ 2 \end{bmatrix} \times_n \begin{bmatrix} 7 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7*7 \\ 3*3 \\ 1*1 \\ 2*2 \end{bmatrix}$$

$$= \begin{bmatrix} 7\times 8 + 7\times 4 \\ 3\times 8 + 3\times 4 \\ 1\times 8 + 1\times 4 \\ 2\times 8 + 2\times 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (0).$$

It is important to make a note if

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

then $x \times_n y \neq y \times_n x$ and

$x \times_n y \neq x$ and $y \times_n x \neq x$

in general.

But for $0 \in C_n$.

$$0 \,.\, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} .\, 0 \,.$$

In view of this one can put forth the following theorem.

Theorem 2.1: Let M = {Collection of all p × q matrices with entries from Z_m; (t, s), t, s ∈ Z_m,$\times_n$} be the groupoid p × q matrix or p × q matrix groupoid. $C_n = 0 < a_1 < a_2 < \ldots < a_{n-2} < 1$ be the chain lattice of *order n. C_nM be the groupoid matrix semiring. C_nM is a non associative semiring of finite order which has no zero divisors.*

Proof: Follows from the fact $a_i \cap a_j \neq 0$ if $a_i \neq 0$ and $a_j \neq 0$. So C_nM has no zero divisors. Further $o(M) < \infty$ and $o(C_n) = n$ so C_nM is a finite semiring which is non associative.

We can not call it semidivision ring or so as it has no unit and is non associative.

Example 2.3: Let N = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z_{10}, (5, 6), \times_n\}$ be matrix groupoid. $C_7 = 0 < a_1 < a_2 < a_3 < \ldots < a_6 < 1$ be the chain lattice of order 7. C_7N be the matrix groupoid semiring.

Let

$$x = a_3 \begin{bmatrix} 8 & 0 \\ 4 & 5 \end{bmatrix} \in C_7N,$$

$$x \times_n x = a_3 \begin{bmatrix} 8 & 0 \\ 4 & 5 \end{bmatrix} \times_n a_3 \begin{bmatrix} 8 & 0 \\ 4 & 5 \end{bmatrix}$$

$$= a_3 \cap a_3 \begin{bmatrix} 8 & 0 \\ 4 & 5 \end{bmatrix} \times_n \begin{bmatrix} 8 & 0 \\ 4 & 5 \end{bmatrix}$$

$$= a_3 \begin{bmatrix} 8 * 8 & 0 * 0 \\ 4 * 4 & 5 * 5 \end{bmatrix}$$

$$= a_3 \begin{bmatrix} 40 + 48 & 4 \times 0 + 6 \times 0 \\ 20 + 24 & 25 + 30 \end{bmatrix}$$

$$= a_3 \begin{bmatrix} 8 & 0 \\ 4 & 5 \end{bmatrix} = x.$$

Thus x is an idempotent of C_7N.

In view of this the following theorem is proved.

Theorem 2.2: Let N= {Collection of all p×q matrices with entries from Z_m; (t, s), t,s $\in Z_m, \times_n$} be the p×q matrix groupoid. $C_n = 0 < a_1 < a_2 < \ldots < a_{n-2} < 1$ be the chain lattice of order n. C_nN be the groupoid matrix semiring. Every $x \in C_nN$ with $|supp x| = 1$ is an idempotent of C_nN if and only if $t + s \equiv 1 \pmod m$.

Proof: First by support of α we mean the number of non zero terms in $\alpha \in C_nN$. So if $|supp\ \alpha| = 1$ then α has only one term.

Given C_nN is non associative semiring. Let x be any element in C_nN with support of x to be 1 then

$$x = a_j \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ a_{21} & \cdots & a_{2q} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pq} \end{bmatrix} \in C_nN; \ a_j \in C_n.$$

$$x \times_n x = a_j \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ a_{21} & \cdots & a_{2q} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pq} \end{bmatrix} \times_n a_j \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ a_{21} & \cdots & a_{2q} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pq} \end{bmatrix}$$

$$= a_j \cap a_j \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ a_{21} & \cdots & a_{2q} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pq} \end{bmatrix} \times_n \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ a_{21} & \cdots & a_{2q} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pq} \end{bmatrix}$$

$$= a_j \begin{bmatrix} a_{11} * a_{11} & a_{12} * a_{12} & \cdots & a_{1q} * a_{1q} \\ a_{21} * a_{21} & a_{22} * a_{22} & \cdots & a_{2q} * a_{2q} \\ \vdots & \vdots & & \vdots \\ a_{p1} * a_{p1} & a_{p2} * a_{p2} & \cdots & a_{pq} * a_{pq} \end{bmatrix}$$

$$= a_j \begin{bmatrix} ta_{11} + sa_{11} & ta_{12} + sa_{12} & \cdots & ta_{1q} + sa_{1q} \\ ta_{21} + sa_{21} & ta_{22} + sa_{22} & \cdots & ta_{2q} + sa_{2q} \\ \vdots & \vdots & & \vdots \\ ta_{p1} + sa_{p1} & ta_{p2} + sa_{p2} & \cdots & ta_{pq} + sa_{pq} \end{bmatrix}$$

$$= a_j \begin{bmatrix} (t+s)a_{11} & (t+s)a_{12} & \dots & (t+s)a_{1q} \\ (t+s)a_{21} & (t+s)a_{22} & \dots & (t+s)a_{2q} \\ \vdots & \vdots & & \vdots \\ (t+s)a_{p1} & (t+s)a_{p2} & \dots & (t+s)a_{pq} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pq} \end{bmatrix}$$

if and only if $t + s \equiv 1 \pmod m$).
Hence the result.
This will be illustrated by an example.

Example 2.4: Let

$$M=\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \mid a_i \in Z_9, 1 \le i \le 8, (7, 3); \times_n \right\}$$

be the groupoid matrix. $C_6 = 0 < a_1 < a_2 < a_3 < a_4 < 1$ be the chain lattice of order 6. C_6M is the matrix groupoid semiring.[1-4]
Let

$$x = a_4 \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{bmatrix} \in C_6M;$$

$$x \times_n x = a_4 \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{bmatrix} \times_n a_4 \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{bmatrix}$$

$$= a_4 \cap a_4 \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{bmatrix} \times_n \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{bmatrix}$$

$$= a_4 \begin{bmatrix} 3*3 & 2*2 \\ 1*1 & 4*4 \\ 0*0 & 5*5 \\ 6*6 & 8*8 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 7+3\times 3 & 2\times 7+2\times 3 \\ 1\times 7+1\times 3 & 4\times 7+4\times 3 \\ 0\times 7+0\times 3 & 5\times 7+5\times 3 \\ 6\times 7+6\times 3 & 8\times 7+8\times 3 \end{bmatrix} = a_4 \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{bmatrix} = x.$$

Thus x is an idempotent of C_6M which substantiates the above theorem.

Clearly if in the matrix groupoid M={Collection of all $p \times q$ matrices with entries from Z_m, (t, s), $\times_n$} if the pair (t, s) used in the matrix groupoid M is interchanged to (s, t) in theorem still the result holds good. Infact there are several such matrix groupoids satisfying the conditions of the theorem.

The next interesting factor would be how many such matrix groupoids exist for a fixed m that is for a given Z_m. The answer is $m - 2$ if m is odd or even.[3]

This will be presented by some simple examples.

Example 2.5. Let M={(a_1, a_2, a_3, a_4, a_5) | $a_i \in Z_{18}$, $1 \le i \le 5$, (10, 9), ×} be the matrix groupoid. C_nM has idempotents when |suppx| = 1, $x \in C_nM$.

The number of pairsgiving such matrix groupoids using Z_{18} are (2, 17), (3, 16), (4, 15), (5, 14), (6, 13), (7, 12), (8, 11), (9, 10), (10, 9), (11, 8), (12, 7), (13, 6), (14, 5), (15, 4), (16, 3) and (17, 2).

Thus there exists 16 such matrix

groupoids and m = 18; so m – 2 = 18 – 2 = 16.

Example 2.6. Let

$$M=\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \mid a_i \in Z_{23}, (18, 6), *, 1 \le i \le 10, \times_n\}$$

be the column matrix groupoid. Every $x \in M$ with |suppx| = 1 is an idempotent in M.

Infact there are 21 such groupoids given by the following pairs (22, 2), (21, 3), (20, 4), (19, 5), (18, 6), (17, 7), (16, 8), (15, 9), (14, 10), (13, 11), (12, 12), (11, 13), (10, 14), (9, 15), (8, 16), (7, 17), (6, 18), (5, 19), (4, 20), (3, 21) and (2, 22).

In view of this the following theorem is proved.

Theorem 2.3: Let M= {Collection of all $p \times q$ matrices with entries from Z_m,(t, s), ($t + s \equiv 1 \mod m$), $\times_n$} be the matrix groupoid.

$C_n = 0 < a_1 < a_2 < \ldots < a_{n-2} < 1$ be the chain lattice. C_nM be the matrix groupoid semiring. There are m – 2 number of matrix groupoid semirings which are such that every element $x \in C_nM$ with |suppx| = 1 is an idempotent of C_nM.

Proof: For any given m, one has only m – 2 elements such that $t + s \equiv 1 \pmod m$). For with $t \neq 0$ or 1 and $s \neq 0$ or 1. The other pairs (t, s) with $t+s \equiv 1 \pmod m$ are (2, m – 1), (3, m – 2), …, (m – 1, 2) which number as (m – 2).

If m is odd then m – 1 is even so that one of the pairs is

$$\left(\frac{m+1}{2}, \frac{m+1}{2}\right).$$

If m is even then m – 1 is odd so that (m/2, m + 2/2) is the middle term. For every $x \in C_nM$; |supp x|=1 is such that $x \times_n x = x$.

First this will be illustrated for odd m as in case of odd m, one has a matrix groupoid semiring which is commutative satisfying ab = ba for all $a, b \in C_nM$.

Example 2.7: Let

$$M=\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_i \in Z_9(5,5),\ 1 \le i \le 12,$$

$(5, 5), \times_n\}$

be the matrix groupoid.
$C_{12} = 0 < a_1 < a_2 < \ldots < a_{10} < 1$ be the chain lattice $C_{12}M$ be the matrix groupoid semiring. Let

$$x=a_5\begin{bmatrix} 3 & 4 & 5 \\ 1 & 0 & 2 \\ 7 & 6 & 8 \\ 0 & 7 & 0 \end{bmatrix} \text{ and } y=a_7\begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 5 \\ 0 & 1 & 3 \\ 6 & 0 & 4 \end{bmatrix} \in C_{12}M.$$

Consider

$$x \times_n y = a_5\begin{bmatrix} 3 & 4 & 5 \\ 1 & 0 & 2 \\ 7 & 6 & 8 \\ 0 & 7 & 0 \end{bmatrix} \times_n a_7\begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 5 \\ 0 & 1 & 3 \\ 6 & 0 & 4 \end{bmatrix}$$

$$= (a_5 \cap a_7) \begin{bmatrix} 3 & 4 & 5 \\ 1 & 0 & 2 \\ 7 & 6 & 8 \\ 0 & 7 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 5 \\ 0 & 1 & 3 \\ 6 & 0 & 4 \end{bmatrix}$$

$$= a_5 \begin{bmatrix} 3*0 & 4*3 & 5*2 \\ 1*1 & 0*4 & 2*5 \\ 7*0 & 6*1 & 8*3 \\ 0*6 & 7*0 & 0*4 \end{bmatrix}$$

$$= a_5 \begin{bmatrix} 6 & 8 & 8 \\ 1 & 2 & 8 \\ 8 & 8 & 1 \\ 3 & 8 & 2 \end{bmatrix} \quad \ldots I$$

Consider

$$y \times_n x = a_7 \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 5 \\ 0 & 1 & 3 \\ 6 & 0 & 4 \end{bmatrix} \times_n a_5 \begin{bmatrix} 3 & 4 & 5 \\ 1 & 0 & 2 \\ 7 & 6 & 8 \\ 0 & 7 & 0 \end{bmatrix}$$

$$= (a_7 \cap a_5) \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 5 \\ 0 & 1 & 3 \\ 6 & 0 & 4 \end{bmatrix} \times_n \begin{bmatrix} 3 & 4 & 5 \\ 1 & 0 & 2 \\ 7 & 6 & 8 \\ 0 & 7 & 0 \end{bmatrix}$$

$$= a_5 \begin{bmatrix} 0*3 & 3*4 & 2*5 \\ 1*1 & 4*0 & 5*2 \\ 0*7 & 1*6 & 3*8 \\ 6*0 & 0*7 & 4*0 \end{bmatrix}$$

$$= a_5 \begin{bmatrix} 6 & 8 & 8 \\ 1 & 2 & 8 \\ 8 & 8 & 1 \\ 3 & 8 & 2 \end{bmatrix}. \quad \ldots II$$

This is the way $\times_n$ operation is performed.

Clearly I and II are identical hence $x \times_n y = y \times_n x$ in $C_{12}M$.

Infact one can conclude $C_{12}M$ is a commutative matrix groupoid semiring as $t = s = 5$.

In view of this the following theorem is proved.

*Theorem 2.4: Let M = {Collection of all $p \times q$ matrices with entries from Z_m, (t,s), *; $t = s$ with $t + s \equiv 1 (\text{mod } m)$ and m is an odd number, $\times_n$} be the matrix groupoid. $C_n = 0 < a_1 < a_2 < \ldots < a_{n-2} = 1$ be the chain lattice. C_nM be the matrix groupoid semiring.*

C_nM is commutative and is a non associative semiring such that every $x \in C_nM$ with $|suppx| = 1$ is an idempotent in C_nM.

Proof: Follows from the following facts. M is a commutative matrix groupoid if and only if $t = s^2$.

If $t = s$ with $t + s = 1$ (mod m) then every $x \in C_nM$ with $|suppx| = 1$ is such that $x \times_n x = x^2 = x^2$.

Now having seen some of the properties satisfied by matrix groupoid semirings C_nM; we proceed on to define other properties like subsemirings, ideals and so on.

First this will be described by some examples.

Example 2.8: Let M = {All 3×4 matrices with entries from Z_8 (2, 4), $\times_n$} be the matrix groupoid. $C_{13} = 0 < a_1 < a_2 < \ldots < a_{11} < 1$ be the chain lattice of order 13. $C_{13}M$

be the matrix groupoid semiring.

$C_{13}M$ has subsemirings.

For take P = {All 3 × 4 matrices with entries from {0, 3, 2, 4, 6} = T ⊆ Z_8 (2, 4), $\times_n$} ⊆ M; P is a subgroupoid of M. Clearly $C_{13}P$ is a matrix subgroupoid semiring, hence a subsemiring which is non associative.

Consider

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_1 \in Z_8, (2, 4), \times_n \right\}$$

be the matrix subgroupoid of M.

$C_{13}P_1$ is a subsemiring of $C_{13}M$.

$$P_{10} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{10} & 0 & 0 \end{bmatrix} \mid a_{10} \in Z_8, \times_n \right\};$$

$C_{13}P_{10}$ is a subsemiring of $C_{13}M$.

It can be easily proved all matrix groupoid semirings has subsemirings. In view of this the following theorem is true[2].

*Theorem 2.5: Let M be any p × q matrix groupoid with entries from G = {Z_n, *, (t, u), $\times_n$}. C_m a chain lattice. C_mM be the matrix groupoid semiring. C_mG has at least ($p \times q\ C_1 + p \times q\ C_2 + \ldots + p \times q\ C_{(p \times q) - 1}$) number of nontrivial subsemirings which are not ideals of C_mM.*

Proof. Let

$$N^1 = \left\{ \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \mid a_1 \in Z_n \right\} \subseteq M$$

then C_mN^1 is a matrix groupoid subsemiring. Thus

$$N^i = \left\{ \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & a_i & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \mid 1 \le i \le p \times q \right\}$$

is such that C_mN^i is a matrix groupoid subsemiring.

Likewise one have the above said number of subgroupoids in M which clearly contributes to $p \times q\ C_1 + p \times q\ C_2 + \ldots + p \times q\ C_{(p \times q) - 1}$ number of matrix groupoid subsemirings in C_mM. Clearly P × A for any A ∈ M and P ∈ N^i is not in N^i unless A ∈ N^i. Hence C_nN^i's are not ideals only subsemirings, $1 \le i \le (p \times q)$.

Now condition for matrix groupoid semirings to have ideals will be discussed.

First some examples in this direction are given.

Example 2.9: Let G= {Z_{18}, (0, 2), $\times_n$} be the groupoid. M = {All 5 × 5 matrices with entries from G under (0, 2), $\times_n$} be the matrix groupoid. LM be the matrix groupoid semiring where L=C_{12} be the chain lattice. Let P = {All 5 × 5 matrices with entries from W = {0, 2, 4, 6, 8, 10, 12, 14, 16}, (0, 2), $\times_n$} ⊆ M be the ideal of M. Clearly LP is the matrix groupoid ideal of LM.

Example 2.10: Let M = {Collection of all 3 ×2 matrices with entries from G = {Z_{12}, *, (2, 2)}, (2, 2), $\times_n$} be the matrix groupoid.

$C_3 = 0 < a_1 < 1$ be the chain lattice. C_3M be the matrix groupoid semiring. Let P = {All 3×2 matrices with entries from {0, 2, 4, 6, 8, 10}, $\times_n$, (2, 2)} $\subseteq$ M be an ideal of M.

Suppose

$$x = a_1 \begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 0 & 8 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 4 & 0 \end{bmatrix} \in C_3M.$$

$$x \times_n y = a_1 \begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 0 & 8 \end{bmatrix} \times_n \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 4 & 0 \end{bmatrix}$$

$$= (a_1 \cap 1) \begin{bmatrix} 2*2 & 3*4 \\ 5*6 & 7*8 \\ 0*4 & 8*0 \end{bmatrix}$$

$$= a_1 \begin{bmatrix} 8 & 2 \\ 10 & 6 \\ 8 & 4 \end{bmatrix} \in P.$$

It is easily verified; C_3P is a matrix groupoid ideal of C_3M.

In view of all these the following theorem is proved.

*Theorem 2.5: Let M = {Collection of all p × q matrices with entries from G = {Z_n, (t, s), *}, (t, s), $\times_n$} be the matrix groupoid. L = C_m be any chain lattice. C_mM be the matrix groupoid semiring. C_mM has ideals if the groupoid G has ideals.*

Proof: Follows from the fact if G has an ideal P then W = {Collection of all p × q matrices with entries from P, (t, s), $\times_n$} $\subseteq$ M is also an ideal of M. Hence C_mW is an ideal of C_mM. Hence the theorem.

In the next result it is shown that there exists a class of matrix groupoid semirings which has ideals.

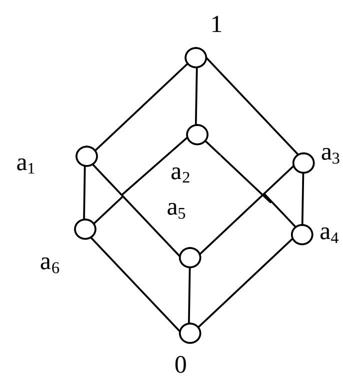
*Theorem 2.6: Let M = {All p × q matrices with entries from G = {Z_n, (t, t), *}, (t, t), $\times_n$} be the matrix groupoid. L = C_m be a chain lattice LM has ideals of the form LP (P $\subseteq$ M) if t/n.*

Proof: G has subgroupoids which are ideals if t/n ^{2}. So M has ideals only if t/n. If P is any such ideal of M then LP will be the ideal of the matrix groupoid semiring.

Replacing chain lattices by finite distributive lattices still the results hold good. However a few examples to that effect is given.

Example 2.11: Let M = {All 2 × 8 matrices with entries from {Z_{12}, *, (6, 2)}, $\times_n$, (6, 2)} be the matrix groupoid.

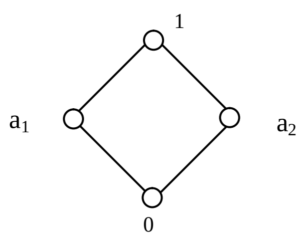
L =



be the distributive lattice. LM is the matrix groupoid semiring which has many zero divisors.

Example 2.12. Let M = {Collection of all 5×1 matrices with entries from G= $\{Z_{15}$, *, (3, 3)}, (3, 3), $\times_n\}$ be the matrix groupoid.

L =



be the distributive lattice LM be the matrix groupoid semiring. LM has ideals as well as zero divisors.

$$P=\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in \{0, 3, 6, 9, 12\}, *, (3, 3) 1 \le i \le 5\right\} \subseteq M$$

is a matrix ideal of the matrix groupoid M. LP $\subseteq$ LM is the matrix ideal of the matrix groupoid semiring.

3. Conclusion

In this paper for the first time matrix groupoid semirings using distributive lattices as semirings is carried out. Several interesting properties enjoyed by these groupoid matrix semirings is obtained.

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