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On the approximation by (N, p_m, q_n) means Of double Fourier series

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Abstract

Generalising an earlier result of Hasegawa[1], degree of approximation of a function belonging to Lip (α, β) class by double Nörlund summability of its double Fourier series has been determined.

Introduction

For $f \in$ Lipα, the degree of approximation by Cesàro mean and by Nörlund means of the Fourier series of f have been studied by Alexits[1], Sahney and Goel[10], Chandra, Quraeshi[6,7] and Quraeshi and Neha[8]. Hasegawa[4] determined the degree of approximation of a continuous function f (x, y) belonging to class lip (α, β) where 0<α<1 and 0<β<1 by double Cesàro means of its double Fourier series. But till now no work seems to have been done to obtain the degree of approximation of the function f(x, y) ∈ Lip (α, β) by double Nörlund summability means (N, p_m, q_n) which includes as special cases, the methods of (H, 1, 1) and (C, 1, 1) summability. In an attempt to make an advanced study in this direction, establish a theorem on the approximation by (N, p_m, q_n) means of double Fourier series.

Definition and Notations:

Let f (x, y) be a Lebesgue integrable function of period 2π with respect to each variable x and y. Let the double Fourier series of f (x, y) be

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n} [a_{mn}\cos mx \cos nx + b_{mn}\cos mx \sin ny + c_{mn}\sin mx \cos ny + d_{mn}\sin mx \sin ny]$$

Where, $\lambda_{m,n} = \begin{cases} \frac{1}{4}, m = n = 0 \\ \frac{1}{2}, m > 0, n = 0 \text{ or } m = 0, n > 0 \\ 1, m > 0, n > 0 \end{cases}$

$$a_{m,n} = 1/\pi^2 \int_Q\int f(x,y) \cos mx \cos ny \, dx \, dy$$

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with similar expression for $b_{m,n}$, $c_{m,n}$ and $d_{m,n}$, where Q denotes the fundamental rectangle (π,- π; π, -π).

A function $f(x, y)$ is said to belong to lip (α, β), if

$$f(x+t, y+s) - f(x, y) = O(|t|^{\alpha} + |s|^{\beta}),$$

uniformly in the point (x, y) as t and s tends to 0 (zero) independently of each other, where $0 < \alpha < 1$, $0 < \beta < 1$. If the capital order 'O' is replaced by little order 'o' in the above definition then f (x, y) is said to belong to lip (α, β).

Let $\{p_m\}$ and $\{q_n\}$ be two sequences of real constants and let

$$P_m = p_0 + p_1 + p_2 + \ldots + p_m,\ P_{-1} = p_{-1} = 0,\ p_0 > 0$$

and $Q_n = q_0 + q_1 + q_2 + \ldots + q_n$, $Q_{-1} = q_{-1} = 0$, $q_0 > 0$

The double Fourier series $\Sigma\, U_{m,n}$ with the sequence of partial sum $\{S_{m,n}\}$ is said to be summable by double Nörlund method, or summable (N, p_m, q_n) if $t_{m,n}$ tends to a limit S as $(m, n) \to \infty$, where the (N, p_m, q_n) means[5], $t_{m,n}$ is defined by

$$t_{m,n} = \frac{1}{P_m\, Q_n} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} p_{m-\mu} q_{n-\nu} s_{\mu-\nu}$$

$$= \frac{1}{P_m\, Q_n} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} p_{\mu} q_{\nu} s_{m-\mu, n-\nu}$$

The necessary and sufficient condition for the regularity of (N, p_m, q_n) are

$$p_m q_n = o(P_m Q_n)$$

and $\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} p_{\mu}\, q_{\nu} = o(|P_m||Q_n|)$, as $(m, n) \to \infty$

Throughout this paper we shall take (N, p_m, q_n) as regular method.

Particular Cases :

There are two important particular cases of (N, p_m, q_n) summability:

1. (C, 1, 1) summability[3], if $p_m = 1$, for every m and $q_n = 1$ for every n.
2. (H, 1, 1) summability[9], if $p_m = 1/m+1$, for every m and $q_n = 1/n+1$, for every n.

We write, $\phi(t, s) = \phi(\mu, \nu; t, s)$

$$= \frac{1}{4}[f(x+t, y+s) + f(x-t, y-s) + f(x-t, y+s) + f(x+t, y-s) - 4f(x, y)]$$

$$S_{m,n}(x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+t,\ y+s)\, K_m(t)\, K_n(s)\, dt\, ds$$

$$\text{Where, } K_m(t) = \frac{2}{m+1} \left(\frac{\sin\frac{1}{2}(m+1)t}{2 \sin\frac{1}{2}t} \right)^2$$

$$K_n(s) = \frac{2}{n+1} \left(\frac{\sin\frac{1}{2}(n+1)s}{2 \sin\frac{1}{2}s} \right)^2$$

$$N_m^p(t) = \frac{1}{2\pi\ Pm} \sum_{\mu=0}^{m} \frac{p_\mu \sin\left(m-\mu+\frac{1}{2}\right) t}{\sin\frac{t}{2}}$$

$$N_n^q(s) = \frac{1}{2\pi\ Qn} \sum_{\nu=0}^{n} \frac{q_\nu \sin\left(n-\nu+\frac{1}{2}\right) s}{\sin\frac{s}{2}}$$

Known Theorem: Hasegawa[4] has proved the following theorem:

Theorem: If a continuous function $f(x,y)$ of period 2π with respect to each x and y belonging to lip (α,β) where $0 < \alpha < 1$ and $0 < \beta < 1$, then

$$S_{m,n}(x, y) - f(x, y) = o\left(m^{-\alpha} + n^{-\beta}\right)$$

uniformly in (x, y) as m and n tend to infinity independently of each other.

If = $\alpha = \beta = 1$, *then*

$$S_{m,n}(x,y) - f(x,y) = o(m^{-1} + n^{-1})$$

uniformly in (x, y) as m, n→∞ independently. These results are also valid for lip (α,β) class.

Main Theorem: The object of the present paper is to extend the above theorem for double N*örlund* summability (N, p_m, q_n) in two directions. In fact we shall establish following theorem.

Theorem : Let (N, p_m, q_n) be regular double N*örlund* method, defined by real, non negative monotonic, non-increasing sequence of coefficients $\{p_m\}$ and $\{q_n\}$ such that $p_m \to \infty$ as m→∞ and $Q_n \to \infty$ as n→∞. If a continuous function f(x, y) of period 2π, with respect to each of x and y belongs to Lip (α, β), *i.e.*,

$f(x+t, y+s) - f(x, y) = o(|t|^{\alpha} + |s|^{\beta})$, then

$$t_{m,n} - f(x,y) = \begin{cases} O\left(\frac{1}{m^{\alpha}} + \frac{1}{n^{\beta}}\right), & 0 < \alpha < 1,\ 0 < \beta < 1 \\ O\left(\frac{\log(m\pi e)}{m} + \frac{\log(n\pi e)}{n}\right), & \alpha = \beta = 1 \end{cases}$$

uniformly in (x, y) as m and n tends to infinity independent of each other where

$$t_{m,n} = \frac{1}{P_m Q_n}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_{m-\mu}\, q_{n-\nu}\, s_{\mu,\nu}$$

$$= \frac{1}{P_m Q_n}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_{\mu}\, q_{\nu}\, s_{m-\mu,n-\nu}$$

is (N, p_m, q_n) means of double Fourier series.

Lemma: Our proof of the theorem needs some lemmas:

Lemma (1) : Tripathi and lal [11]

(i) If $0 \le t < 1/m$, then

$$N_m^{(p)}(t) = O[m].$$

(ii) For $0 \le s \le 1/n$ *then*

$$N_n^{(q)}(s) = O[n].$$

Lemma(2): Tripathi and Lal[12]

(i) If $1/m \le t < \pi$, and any m

$$N_m^{(p)}(t) = O\left[\frac{P[1/t]}{tP_m}\right]$$

(ii) For $1/n \le s \le \pi$ *and any n*

$$N_n^{(q)}(s) = O\left[\frac{Q[1/s]}{sQ_n}\right]$$

Proof of the Main Theorem:

The $(\mu,\nu)^{th}$ partial sum of the double Fourier series at (t,s) = (x,y) is given by

$$S_{\mu,\nu} - f(x,y) = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{\phi(t,s)\sin\left(\mu+\frac{1}{2}\right)t\,\sin\left(\nu+\frac{1}{2}\right)s}{\sin\frac{t}{2}\sin\frac{s}{2}}\,dtds$$

then

$$\frac{1}{P_m Q_n}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_{\mu}q_{\nu}\left[S_{m-\mu,n-\nu} - f(x,y)\right]$$

$$= \int_0^{\pi}\int_0^{\pi}\phi(t,s)\frac{1}{2\pi P_m}\sum_{\mu=0}^{m} p_{\mu}\frac{\sin\left(m-\mu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

$$\frac{1}{2\pi Q_n}\sum_{\nu=0}^{n} q_{\nu}\frac{\sin\left(n-\nu+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,dtds$$

or

$$t_{m-n} - f(x,y) = \int_0^{\pi}\int_0^{\pi}\phi(t,s)N_m^{(p)}(t)N_n^{(q)}(s)\,dtds$$

$$=\left[\int_0^{1/m+1}\int_0^{1/n+1} + \int_0^{1/m+1}\int_{1/n+1}^{\pi} + \int_{1/m+1}^{\pi}\int_0^{1/n+1} + \int_{1/m+1}^{\pi}\int_{1/n+1}^{\pi}\right]\phi(t,s)N_m^{(p)}(t)N_n^{(q)}(s)dtds.$$

$$= I_1 + I_2 + I_3 + I_4, \qquad (1)$$

Now,

$$I_1 = \int_0^{1/m+1}\int_0^{1/n+1}\phi(t,s)N_m^{(p)}(t)N_n^{(q)}(s)dtds$$

$$= O\left[(m+1)(n+1)\int_0^{1/m+1}\int_0^{1/n+1}(t^\alpha + s^\beta)dtds\right] \quad \text{by lemma [1]}$$

$$= O\left[(m+1)(n+1)\int_0^{1/m+1} t^\alpha \left\{\int_0^{1/n+1} ds\right\} dt\right]$$

$$+ O\left[(m+1)(n+1)\int_0^{1/n+1} s^\beta \left\{\int_0^{1/m+1} dt\right\} ds\right]$$

$$= O\left[(m+1)(n+1)\int_0^{1/m+1} t^\alpha \frac{1}{n+1} dt\right] + O\left[\frac{(m+1)(n+1)}{m+1}\int_0^{1/n+1} s^\beta\, ds\right]$$

$$= O\left[(m+1)\frac{1}{(\alpha+1)}t^{\alpha+1}\right]_0^{1/m+1} + O\left[(n+1)\frac{1}{(\beta+1)}s^{\beta+1}\right]_0^{1/n+1}$$

$$= O\left[\frac{1}{(m+1)^\alpha}\right] + O\left[\frac{1}{(n+1)^\beta}\right] = O\left[\frac{1}{(m+1)^\alpha} + \frac{1}{(n+1)^\beta}\right] \qquad (2)$$

Next, $I_2 = \int_0^{1/(m+1)}\int_{\frac{1}{(n+1)}}^{\pi}\phi(t,s)N_m^{(p)}(t)N_n^{(q)}(s)dtds$

$$= O\left[\left[\frac{(m+1)}{Q_n}\int_0^{1/(m+1)}\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s}(t^\alpha + s^\beta)dtds\right]\right] \quad \text{by lemma(2)}$$

$$= O\left[\frac{(m+1)}{Q_n}\int_0^{1/(m+1)}\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s}t^\alpha dtds\right] + O\left[\frac{(m+1)}{Q_n}\int_0^{1/(m+1)}\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s}s^\beta dtds\right]$$

$= I_{2.1} + I_{2.2}$, say (3)

$$I_{2.1} = O\left[\frac{(m+1)}{Q_n}\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s}\{\int_0^{1/(m+1)} t^{\alpha}dt\}ds\right]$$

$$= O\left[\frac{(m+1)}{Q_n}\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s}\{\frac{1}{(\alpha+1)}t^{\alpha+1}\}_0^{1/(m+1)}ds\right]$$

$$=O\left[\frac{(m+1)}{Q_n}\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s(\alpha+1)}.\frac{1}{(m+1)^{\alpha+1}}ds\right]$$

$$=O\left[\frac{1}{Q_n(m+1)^{\alpha}}\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s}.ds\right]$$

$$= O\left[\frac{1}{Q_n(m+1)^{\alpha}}\int_{1/(n+1)}^{\pi}-\frac{Q(u)}{u}.du\right]$$

$$= O\left[\frac{1}{Q_n(m+1)^{\alpha}}\frac{Q_n}{(n+1)}\int_1^{(n+1)}ds\right]$$

$$= O\left[\frac{1}{(n+1)(m+1)^{\alpha}}.(n)\right]$$

$$= O\left(\frac{1}{n(m+1)^{\alpha}}\right)(n)$$

$$I_{2.1} = O(1/(m+1)^{\alpha}) = O(1/(m)^{\alpha}) \quad (4)$$

$$I_{2.2} = O\left[\frac{(m+1)}{Q_n}\int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^{\pi}\frac{Q(\frac{1}{s})}{s}s^{\beta}dtds\right]$$

$$=O\left[\frac{(m+1)}{Q_n}\int_{\frac{1}{(n+1)}}^{\pi}\frac{Q(\frac{1}{s})}{s}s^{\beta}\{\int_0^{1/(m+1)}dt\}ds\right]$$

$$=O\left[\frac{(m+1)}{(m+1).Q_n}.\int_{\frac{1}{(n+1)}}^{\pi}\frac{Q(\frac{1}{s})}{s}s^{\beta}ds\right]$$

$$=O\left[\frac{1}{Q_n}.\int_{\frac{1}{\pi}}^{(n+1)}\{\frac{-Q(u)}{u}\}\frac{1}{u^{\beta}}\ du\right]$$

$$= O\left[\frac{1}{Q_n}.\frac{Q_n}{(m+1)}\int_{\frac{1}{\pi}}^{(n+1)}\frac{1}{u^{\beta}}\ du\right]$$

$$= \begin{cases} o\left[\frac{1}{n+1}\log u\right], \ \beta=1 \\ o\left[\frac{1}{n+1}\left\{\frac{1}{(-\beta+1)u^{\delta-1}}\right\}\right], \beta\neq 1 \end{cases}$$

$$= \begin{cases} O\left[\frac{\log n\pi}{n}\right], \beta=1 \\ O[\frac{1}{n}\{\frac{1}{n^{\beta-1}}-\frac{1}{\left(\frac{1}{\pi}\right)^{\beta-1}}\}], \beta\neq 1 \end{cases}$$

$$I_{2.2} = \begin{cases} O\left[\frac{\log n\pi}{n}\right], \beta=1 \\ O(\frac{1}{n^{\beta}}), 0<\beta<1 \end{cases} \quad (5)$$

By (3),(4) and (5), we have

$$I_2 = \begin{cases} O(\frac{1}{m^{\alpha}} + \frac{1}{n^{\beta}}), 0 < \alpha < 1, 0 < \beta < 1 \\ O\left(\frac{1}{m} + \frac{\log(n\pi)}{n}\right), \alpha = 1, \beta = 1 \end{cases} \quad (6)$$

Similarily, $I_3 = \begin{cases} O(\frac{1}{m^{\alpha}} + \frac{1}{n^{\beta}}), 0 < \alpha < 1, 0 < \beta < 1 \\ O\left(\frac{1}{n} + \frac{\log(m\pi)}{m}\right), \alpha = 1, \beta = 1 \end{cases}$ (7)

Lastly, $I_4 = \int_{1/(m+1)}^{\pi}\int_{1/(n+1)}^{\pi} \frac{P(\frac{1}{t})Q(\frac{1}{s})}{tP_m.sQ_n}[t^{\alpha} + s^{\beta}]dtds$

$$= O\left[\frac{1}{P_mQ_n}\int_{1/(m+1)}^{\pi}\frac{P(1/t)}{t}t^{\alpha}\{\int_{1/(n+1)}^{\pi}\frac{Q(1/s)}{s}ds\}dt\right]$$

$$+ O\left[\frac{1}{P_mQ_n}\int_{1/(m+1)}^{\pi}\frac{Q(1/s)}{s}s^{\beta}\{\int_{1/(n+1)}^{\pi}\frac{P(1/t)}{t}dt\}ds\right]$$

$$= I_{4.1} + I_{4.2} \quad (8)$$

$$I_{4.1} = O\left[\frac{1}{P_mQ_n}\int_{1/(m+1)}^{\pi}\frac{P(\frac{1}{t})}{t}t^{\alpha}\{\int_{1/(n+1)}^{\pi}\frac{Q(\frac{1}{s})}{s}ds\}dt\right]$$

$$= O\left[\frac{1}{P_mQ_n}\int_{1/(m+1)}^{\pi}\frac{P(\frac{1}{t})}{t}t^{\alpha}\{\int_{1/(n+1)}^{\pi}\frac{Q(u)}{u}du\}dt\right]$$

$$= O\left[\frac{1}{P_mQ_n}\int_{1/(m+1)}^{\pi}\frac{P(\frac{1}{t})}{t}t^{\alpha}\{\frac{Q_n}{(n+1)}\int_{1/\pi}^{(n+1)}du\}dt\right]$$

$$= O\left[\frac{1}{(n+1)P_m}\int_{1/(m+1)}^{\pi}\frac{P(\frac{1}{t})}{t}t^{\alpha}(n - \frac{1}{\pi})dt\right]$$

$$= O\left[\frac{1}{P_m}\int_{1/(m+1)}^{\pi}\frac{P(\frac{1}{t})}{t}t^{\alpha}dt\right]$$

$$= O\left[\frac{1}{P_m}\int_{1/\pi}^{(m+1)}\frac{P(u)}{u}\frac{1}{u^{\alpha}}du\right] \text{ as } I_{2.2}$$

$$= O\left[\frac{1}{P_m}\frac{P_m}{(m+1)}\int_{1/\pi}^{(m+1)}\frac{1}{u^{\alpha}}du\right] =$$

$$= \begin{cases} O\left[\frac{1}{n}\{\log u\}_{1/\pi}^{n}\right], \alpha = 1 \\ O\left[\frac{1}{m}\{\frac{1}{m^{\alpha-1}} + \pi^{\alpha-1}\}\right], \alpha \neq 1 \end{cases}$$

$$= \begin{cases} O\left[\log\frac{m\pi}{m}\right], \alpha = 1 \\ O\left[\frac{1}{m^{\alpha}}\right], \alpha \neq 1 \end{cases}$$

Similarly, $I_{4.2} = \begin{cases} O\left[\log\frac{ne}{n}\right], \beta = 1 \\ O\left[\frac{1}{n^{\beta}}\right], \beta \neq 1 \end{cases}$ (9)

By (8) and (9), $I_4 = \begin{cases} O\left(\frac{1}{m^{\alpha}}+\frac{1}{n^{\beta}}\right), 0<\alpha<1, 0<\beta<1 \\ O\left(\frac{\log(m\pi e)}{m}\right) + O\left(\frac{\log(n\pi e)}{n}\right), \alpha=\beta=1 \end{cases}$ (10)

From (1), (2), (6), (7) and (10), we have

$$t_{m,n} - f(x, y) = \begin{cases} O\left(\frac{1}{m^{\alpha}}+\frac{1}{n^{\beta}}\right), 0<\alpha<1, 0<\beta<1 \\ O\left(\frac{1}{m}+\frac{\log(m\pi)}{m}\right) + O\left(\frac{1}{n}+\frac{\log(n\pi)}{n}\right), \alpha=\beta=1 \end{cases}$$

$$= \begin{cases} O\left(\frac{1}{m^{\alpha}}+\frac{1}{n^{\beta}}\right), 0<\alpha<1, 0<\beta<1 \\ O\{\left(\frac{1+\log(m\pi)}{m}\right) + \left(\frac{1+\log(n\pi)}{n}\right)\}, \alpha=\beta=1 \end{cases}$$

Thus, $t_{m,n} - f(x, y) = \begin{cases} O\left(\frac{1}{m^{\alpha}}+\frac{1}{n^{\beta}}\right), \\ O\left(\frac{\log(m\pi e)}{m}+\frac{\log(n\pi e)}{n}\right) \end{cases}$

This completes the proof of the theorem.

Remarks :

1. If $f \in$ Lip (α, β), $0 < \alpha < 1$, $0 < \beta < 1$, $P_{\mu} = \frac{1}{m+1}$, $\forall\ \mu$; $Q_{\nu} = \frac{1}{n+1}$, $\forall\ \nu$ then the result of Hasegawa[4] becomes the particular case of our theorem.

2. The result of our theorem are also valid if $f(x, y) \in$ lip(α, β)

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