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Groupoid Semirings Using Chain Lattices as Semirings

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Abstract

In this paper for the first time groupoid semirings using chain lattices as semirings are introduced and their properties are analysed. Here chain lattices are used as semirings and groupoid semirings are non associative semirings of finite order, as both the groupoids G and the chain lattice are of finite order.

Key words: Groupoid semirings, chain lattice, groupoid, pseudo zero divisors, subsemirings.

1. Introduction

In this paper a study of groupoid semirings using chain lattices as semirings is carried out. At the outset groupoids do not have identity element and as all groupoids considered here are only of the form $\{Z_n, *, (t, s)\}$; $0 \in Z_n$ by $0 * g \neq 0$ so this 0 is defined as a pseudo zero of the groupoid.

The basic definition of chain lattices and groupoids can be had from[1,3,4,5]. All chain lattices used are also of finite order. Just for the sake of completeness the definition of groupoid semirings is recalled. It is important to mention at this juncture that groupoids in general may not contain the identity element. So while defining the concept of groupoid semirings some sort of difficulty is experienced. However such study is also interesting and innovative. This paper has three sections. Section one is introductory in nature. Section two give all properties enjoyed by groupoid semirings C_nG. The final section gives conclusions based on our study.

2. Groupoid Semirings using Chain Lattices as Semirings :

Definition 2.1: Let $(G, *)$ be a groupoid of finite order; G may or may not contain identity. Let C_n be a chain lattice of order n. Let C_nG = {Collection of all finite formal sums of the form $\sum_i \alpha_i g_i$; $\alpha_i \in C_n$ and g_i G}. Define two binary operations + and $\times$ on C_nG.

For

$$\alpha = \sum_i \alpha_i g_i \text{ and } \beta = \sum_i \beta_i g_i \in C_nG;$$

$$\alpha + \beta = \sum_i \alpha_i g_i + \sum_i \beta_i g_i$$

$$= \sum_i (\alpha_i + \beta_i) g_i$$

$(\alpha_i, \beta_I \in C_n$ *and* $\alpha_i + \beta_I = \alpha_i \cup \beta_i \in C_n)$

$$= \sum_i (\alpha_i \cup \beta_i) g_i.$$

$$\alpha + 0 = \sum_i \alpha_i g_i + 0$$

$$= \sum_i (\alpha_i \cup 0) g_i$$

$$= \sum_i \alpha_i g_i = \alpha \; ; \; 0 \in C_n.$$

$$\alpha \times \beta = \alpha_i g_i \times \sum_i \beta_i g_i$$

$$= \sum_k \gamma_k g_k$$

where $\gamma_k = \alpha_i \beta_j = \alpha_i \cap \beta_j \in C_n$ *and* $g_k = g_i * g_j \in G$.

Clearly $0\,\alpha = 0 = 0\,\alpha$ *for all* $\alpha \in C_nG$. $0 \in C_n$. $0 = g_0 \in Z_n$ *is defined as the pseudo zero as* $g_0 * g \neq 0$ *for* $g \in G$.

As $1 \in C_n$; $1 \;\; g_i \in C_nG$, *thus* $G \subseteq C_nG$; *however as 1 may or may not be present in G;* $C_n \not\subset C_nG$ *if* $1 \notin G$.

Thus $\{C_nG, +, \times\}$ *is defined as the groupoid semiring of the groupoid G over the semiring* C_n.

The following observations are essential.

i. *C_nG is a non-associative semiring.*
ii. *$C\text{-}_nG$ may or may not contain identity element; C_nG has 1 if and only if $1 \in G$.*
iii. *C_nG is commutative if and only if G is commutative.*
iv. *As 1 C_nG since G has no identity C_nG is never a non-associative semifield.*

Thus C_nG in general is a non-associative, non commutative semiring without identity.

This method enables one to obtain semirings which are non-associative in a non abstract way. Since both C_n and G are of finite order the groupoid semiring C_nG is of finite order.

This situation is represented by the following examples.

Example 2.1: Let $G = \{Z_{10}, *, (2, 6)\}$ be the groupoid. C_6 be the chain lattice; C_6G is the groupoid semiring of G over C_6.

Order of C_6G is of finite order. Let $C_6 = 0 < a_1 < a_2 < a_3 < a_4 < 1$.

Let

$$\alpha = \sum_{i=1}^{10} a_i g_i = a_1 6 + a_2 1 + a_2\, 7$$

and

$$\beta = a_3\, 6 + a_4\, 9 + a_4\, 5 + 1 \;\; 3 \in C_6G.$$

$$\alpha + \beta = a_1 6 + a_2\, 1 + a_2\, 7 + a_3\, 6 + a_4\, 9 + a_4\, 5 + 1\, 3$$

$$= a_2 \;\; 1 + 1{\cdot}3 + a_4\, 9 + a_4\, 5 + a_2\, 7 + (a_1 \cup a_3) 6$$

$$= a_2 \;\; 1 + 1{\cdot}3 + a_4\, 9 + a_4\, 5 + a_2\, 7 + a_3\, 6 \in C_6G.$$

$\alpha\beta = \alpha \times \beta$

$= (a_1 6 + a_2\, 1 + a_2\, 7) \times (a_3\, 6 + a_4\, 9 + a_4\, 5 + 1\, 3)$

$= (a_1 \cap a_3)\, (6 * 6) + (a_2 \quad a_3)\, (1 * 6) + (a_2 \cap a_3)\, (7 * 6) + (a_1 \cap a_4)\, 6 * 9 + (a_2 \cap a_4)\, 1 * 9 + (a_2 \cap a_4)\, 7 * 9 + (a_1 \cap a_4)\, 6 * 5 + (a_2 \cap a_4)\, (1 * 5) + (a_2 \cap a_4)\, 7 * 5 + (a_1 \cap 1)\, (6 * 3) + (a_2 \cap 1)\, (1 * 3) + (a_2 \cap 1)\, (7 * 3)$

$= a_1\, (2 \times 6 + 6 \times 6) + a_2\, (2 \times 1 + 6 \times 6) + a_2\, (2 \times 7 + 6 \times 6) + a_1\, (2 \times 6 + 6 \times 9) + a_2\, (1 \times 2 + 9 \times 6) + a_2\, (7 \times 2 + 9 \times 6) + a_1\, (2 \times 6 + 5 \times 6) + a_2\, (1 \times 2 + 5 \times 6) + a_2\, (7 \times 2 + 5 \times 6) + a_1\, (6 \times 2 + 3 \times 6) + a_2\, (1 \times 2 + 3 \times 6) + a_2\, (7 \times 2 + 3 \times 6)$

$= a_1 8 + a_2 8 + a_2\, 0 + a_1 6 + a_2 6 + a_2 8 + a_1 2 + a_2 2 + a_2 4 + a_1\, 0 + a_2\, 0 + a_2 2$

$= (a_1 \cup a_2)8 + 0 + (a_1 \cup a_2)6 + (a_1 \cup a_1)2 + a_1 4$

$= 0 + a_2 8 + a_2 6 + a_2 2 + a_4 4 \in C_6G.$ … I

This is the way + and × operation is performed.

Clearly $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in C_6G$. Thus $(C_6G, +)$ is a commutative semigroup in fact a monoid.

Now $\alpha \neq \beta\alpha$ in general for $\alpha, \beta \in C_6G$. Consider

$\beta\alpha = \beta \times \alpha$

$= (a_3 6 + a_4 9 + a_4 5 + 1\, 3) \times (a_1 6 + a_2\, 1 + a_2 7)$

$= (a_3 \cap a_1)\, 6 * 6 + (a_4 \cap a_1)\, 9 * 6 + (a_4 \cap a_1)\, 5 * 6 + (1 \cap a_1)\, 3 * 6 + (a_3 \cap a_2)\, 6 * 1 + (a_4 \cap a_2)\, 9 * 1 + (a_4 \cap a_2)\, 5 * 1 + (1 \cap a_2)\, 3 * 1 + (a_3 \cap a_2)\, 6 * 7 + (a_4 \cap a_2)\, 9 * 7 + (a_4 \cap a_2)\, 5 * 7 + (1 \cap a_2)\, 3 * 7$

$= a_1 8 + a_1 4 + a_1 6 + a_1 2 + a_2 8 + a_2 4 + a_2 6 + a_2 2 + a_2 4 + a_2 0 + a_2 2 + a_2 8\ (a_2\ a_2 = a_2)$

$= (a_1 \cup a_2)8 + a_2 0 + (a_1 \cup a_2)4 + (a_1 \cup a_2)6 + (a_1 \cup a_2)2$

$= a_2 8 + a_2 0 + a_2 4 + a_2 6 + a_2 2$ …II

I and II are equal so for this α and β;

$\alpha \times \beta = \beta \times \alpha = \alpha\beta = \beta\alpha.$

Let $\alpha = 1.3$ and $\beta = 1.5 \in C_6G;$

$\alpha \times \beta = 1.3 \times 1.5$

$= (1 \cap 1)\, (3 * 5)$

$= 1\, (6 + 30)$

$= 1 \cdot 6$ … I

$\beta \times \alpha = 1.5 \times 1.3$

$= (1 \cap 1)\, (5 \times 2 + 3 \times 6)$

$= 1.8$ …II

I and II are not equal so $\alpha \times \beta \neq \beta \times \alpha$.

It is proved $(\alpha \times \beta) \times \gamma \neq \alpha \times (\beta \times \gamma)$ in general for $\alpha, \beta, \gamma \in C_6G$.

Let $\alpha = a_1 4$, $\beta = a_2 2$ and $\gamma = a_3 5 \in C_6G$.

Let

$\alpha \times (\beta \times \gamma) = a_1\, 4 \times (a_2\ \ 2 \times a_3\, 5)$

$= a_1 4\, (a_1 \cap a_2\, (2 * 5))$

$= a_1 4 * (2 \times 2 + 5 \times 6)$

$= a_1\, (4 * 4)$

$= a_1\, (4 \times 2 + 4 \times 6)$

$= a_1 2$ …I

$(\alpha \times \beta) \times \gamma = (a_1\ 4 \times a_2\, 2) \times a_3\, 5$

$= (a_1 \cap a_2)\, (4 * 2) \times a_3 \cdot 5$

$= a_1\, (2 \times 4 + 2 \times 6) \times a_3 5$

$= (a_1 \cap a_3)\, (0 * 5)$

$= a_1\, (6 \times 5) = a_1\, 0$ …II

I and II are not equal. So $(\alpha \times \beta) \times \gamma \neq \alpha \times (\beta \times \gamma)$ for $\alpha, \beta, \gamma \in C_6G$.

Clearly $a_i0 \neq 0$ for $0 \;\; G$ and $a_i \in C_6$.

Example 2.2: Let $G = \{Z_{11}, (3, 0), *\}$ be the groupoid of order 11. Let $C_5 = \{0 < a_1 < a_2 < a_3 < 1\}$ be the chain lattice. C_5G is the groupoid semiring of the groupoid G over the semiring C_5. This is the way operations are performed on C_5G.

Let

$$\alpha = a_19 + a_210 + a_36 + a_28$$

and

$$\beta = a_25 + a_37 \in C_5G.$$

$$\begin{aligned}\alpha \times \beta &= (a_19 + a_210 + a_36 + a_28) \times (a_25 + a_37) \\ &= a_1 \cap a_2 (9 \times 3 + 0) + a_2 \cap a_2 (10 \times 3 + 0) + (a_3 \cap a_2) (6 \times 3 + 0) + (a_3 \cap a_3) (8 \times 3 + 0) \\ &= a_15 + a_28 + a_27 + a_32 \qquad \ldots \text{I}\end{aligned}$$

$$\begin{aligned}\beta \times \alpha &= (a_25 + a_37) \times (a_19 + a_210 + a_36 + a_28) \\ &= (a_2 \cap a_1) (5 * 9) + (a_3 \cap a_1) (7 * 9) + (a_2 \cap a_2) (5 * 10) + (a_3 \cap a_2) (7 * 10) + (a_2 \cap a_3) (5 * 6) + (a_3 \cap a_3) (7 * 6) + (a_2 \cap a_2) (5 * 8) + (a_3 \cap a_2) (7 * 8) \\ &= a_110 + a_13 + a_210 + a_23 + a_210 + a_310 + a_210 + a_23 \\ &= a_310 + a_23 \qquad \ldots \text{II}\end{aligned}$$

I and II are distinct, so $\alpha \times \beta \neq \beta \times \alpha$ for $\alpha, \beta \in C_5G$.

In C_5G $(\alpha \times \beta) \times \gamma \neq \alpha \times (\beta \times \gamma)$; in general for $\alpha, \beta, \gamma \in C_5G$.

Definition 2.2: Let $G = \{Z_n, *, (t, s)\}$ be the groupoid. C_n be a chain lattice. C_nG be the groupoid semiring. For $\alpha, \beta \in C_nG$ if $\alpha \times \beta = g_0$ in G then $\alpha \times \beta$ is pseudo zero divisor of C_nG.

Theorem 2.1: C_nG be the groupoid semiring. C_nG can have pseudo zero divisors but no zero divisors.

Proof: The result is true from the following example 2.3.

Example 2.3: Let $G = \{Z_6; (0, 3), *\}$ be the groupoid of order 6. $C_7 = \{0 < a_1 < a_2 < a_3 < a_4 < a_5 < 1\}$ be the chain lattice of order 7. C_7G be the groupoid semiring of the groupoid G over the semiring. C_7G is a one sided pseudo zero divisor. $\alpha = 5$ and $\beta = 4 \in C_7G$

$$\begin{aligned}\alpha \times \beta &= 5 * 4 \\ &= (5 * 4) \\ &= g_0 \qquad \ldots \text{I}\end{aligned}$$

$$\begin{aligned}\beta \times \alpha &= 4 * 5 \\ &= (4 \times 0 + 5 \times 3) \\ &= 3 \qquad \ldots \text{II}\end{aligned}$$

$\alpha \times \beta \neq \beta \times \alpha$; $\alpha \times \beta$ gives a one sided pseudo zero divisor but $\beta \times \alpha \neq g_0$.

So groupoid semirings can have left pseudo zero divisors which are not right pseudo zero divisors.

Example 2.4: Let $G = \{Z_{21}, *, (19, 2)\}$ be the groupoid of finite order. $C_{10} = \{0 < a_1 < a_2 < \ldots < a_8 < 1\}$ be the chain lattice. $C_{10}G$ be the groupoid semiring of G over C_{10}. $C_{10}G$ has a collection of pseudo zero divisors.

$T = \{(g, g) \mid g \in Z_{21}\}$ is such that $x * x = g_0$ for all x T.

Suppose $(19, 2)$ in example 2.4 is replaced by the pair $(7, 14)$ can one find more pseudo zero divisors. The answer is yes.

For $\{(3, 3), (3, 15), (3, 18), (3, 6), (3, 9), (3, 12), (6, 15), (6, 18), (6, 6), (6, 9), (6, 12), (9, 15), (9, 18), (9, 6), (9, 12), (9, 9), (12, 15), (12, 3), (12, 6), (12, 12), (12, 9), (12, 18), (15, 3), (15, 6), (15, 12), (15, 9), (15, 15), (15, 18), (18, 3), (18, 6), (18, 9), (18, 12), (18, 15), (18, 18)\} = W \subseteq C_{10}G$ is such that $x \times y = g_0$ for every $(x, y) \in W$; here in the groupoid $(19, 2)$ is replaced by $(7, 14)$. By default of notation represent $(x, y) \in W$ by $a_ix + a_jy$; $a_i, a_j \in C_{10}$.

For instance take $x = (18, 9)$ and $y = (12, 6) \in W$.

Let $x = a_6 18 + a_2 9$ and $y = a_{-1}2 + a_2 6$.

$$
\begin{aligned}
x \times y &= (a_6 18 + a_2 9) \times (a_1 12 + a_2 6) \\
&= (a_6 \cap a_1)(18 * 12) + (a_1 \cap a_2)(9 * 12) + (a_6 \cap a_2)(18 * 6) + (a_2 \cap a_2)(9 * 6) \\
&= a_1 [18 \times 7 + 12 \times 14] + a_1 [9 \times 7 + 12 \times 14] + a_2 (18 \times 7 + 6 \times 14) + a_2 (9 \times 7 + 6 \times 14) \\
&= a_1 (g_0) + (g_0) + a_2 (g_0) + a_2 (g_0) \\
&= (a_1 \cup 1 \cup a_2 \cup a_2)\, g_0 \\
&= g_0.
\end{aligned}
$$

Thus $C_{10}G$ has pseudo zero divisors.

Further in case of $G = \{Z_{21}, *, (7, 14)\}$; $x = a_1 3 + a_2 6 + a_3 9 + a_4 15 + a_5 18 + a_6 12 \in C_{10}G$ is such that $x \times x = \{g_0\}$.

Now if the pair $(7, 14)$ is replaced by $(3, 18)$ then also $C_{10}G$ has many other pseudo zero divisors. But the above mentioned x is not a pseudo zero divisor if $(7, 14)$ is replaced by $(3, 18)$.

Thus if $G = \{Z_m, *, (t, s)\}$; m a non prime $t + s = g_0 = 0 \in Z_{21}$. t and s divisors or m or $(t, m) = d$ and $(t, m) = d$ then C_nG for any chain lattice. C_n has nontrivial pseudo zero divisors which is stated and proved in the following theorem.

Theorem 2.2: Let $G = \{Z_m, * (t, s)\}$ *(*m *a non prime,* $t + s = 0 = g_0$*.* t *and* s *are non primes in* Z_m*) be the groupoid.* C_n *be the chain lattice.* C_nG *be the groupoid semiring.* C_nG *has nontrivial pseudo zero divisors, pseudo right zero divisors or pseudo left zero divisors some of which are also pseudo nilpotents of order two.*

Proof: Given m is a non prime and $t, s \in Z_m$ are non primes such that $t + s = g_0$ (mod m). C_nG is the groupoid semiring of the groupoid G over the semiring C_n.

Let $x = a_ip_1 + a_jp_2 + a_kp_3 + a_rp_4$ and $y = a_qp_5 + a_rp_6 + a_lp_7 + a_op_8 \in C_nG$; such that p_it g_0 mod m and $p_is \equiv g_0$ mod m $(1 \le i \le 8)$.

Clearly $x \times y = y \times x = a_ig_0 = g_0$ if $a_i = a_j = a_k = a_r = a_q = a_l = a_o = 1$. Thus C_nG has pseudo zero divisors.

This is illustrated by some examples.

Example 2.5: Let $G = \{Z_{24}, *, (6, 18)\}$ be the groupoid of order 24. $C_8 = \{0 <$

$a_1 < a_2 < \ldots < a_6 < 1\}$ be the chain lattice and C_8G be the groupoid semiring of the groupoid G over the semirings C_8.

Let $x = a_34 + a_58 + a_612 + a_720$ and $y = \{a_620 + a_712 + a_78\} \in C_8G$.

Clearly $x \times y = g_0$ also $y \quad x = g_0$. Further $x \times x = g_0$ and $y \times y = g_0$ are pseudo zero divisors of C_nG.

Hence the claim in the theorem is true in case of this groupoid semiring C_8G.

Next the condition for these groupoid semirings to contain idempotents is first illustrated by some examples and then the essential related theorem is given.

Example 2.6: Let $G = \{Z_{10}, *, (5, 6)\}$ be the groupoid. $C_6 = \{0 < a_1 < a_2 < a_3 < a_4 < 1\}$ be the chain lattice. C_6G be the groupoid semiring.

Let $x = a_16 \in C_6G$.

$$x \times x = a_16 \times a_16$$
$$= (a_1 \cap a_1)(6 * 6) = a_16 = x.$$

Thus x is an idempotent in C_6G. Every $t \in Z_{10}$ is an idempotent in C_6G. In fact a_i1, $a_i2, a_i3, \ldots, a_i9 \in C_6G; 1 \leq i \leq 4$ and $a_i = 1$ are all idempotents of C_6G. Thus there are 45 such idempotents.

Consider

$$a_43 \times a_43 = (a_4 \cap a_4)(3 \times 5 + 3 \times 6)$$
$$= a_4 (15 + 18)$$
$$= a_43$$

is again idempotent of C_6G.

In fact $0 = g_0 \in C_6G$ is termed as the trivial idempotent.

Hence $B = \{a_in \mid a_i = 1$ or $1 \leq i \leq 4$ and $n = 0, 1, \ldots, 9\} \subseteq C_6G$ is just an idempotent groupoid under product. However B is not closed under addition. In fact B under '+' will generate the whole of C_6G.

Example 2.7: Let $G=\{Z_{13}, (4, 10),*\}$ be the groupoid of order 13 and $C_5 = \{0 < a_1 < a_2 < a_3 < 1\}$ be the chain lattice of order 5. C_5G be the groupoid semiring of groupoid G over the semiring C_5. $B = \{a_jg_i \mid g_i \in Z_{13}, a_j \in C_5\}$ are all idempotents of C_5G.

Let $x = a_37 \in C_5G$,

$$x \times x = a_37 \times a_37$$
$$= a_3 \cap a_3 (7 * 7)$$
$$= a_3 (7 \times 4 + 7 \times 10)$$
$$= a_37.$$

Let $x = a_26$ and $y = a_110 \in B$.

$$x \times y = a_26 \times a_110$$
$$= a_2 \cap a_1 (6 * 10)$$
$$= a_1 (6 \times 4 + 10 \times 10)$$
$$= a_17 \in B.$$
$$y \times x = a_110 \times a_26$$
$$= a_1 \cap a_2 (10 * 6)$$
$$= a_1 (10 \times 4 + 6 \times 10)$$
$$= a_13 \in B.$$

Clearly $x \times y \neq y \times x$ thus B is a non commutative idempotent groupoid semiring of order 49. However B will generate whole of C_5G under $+$.

Here a characterization of groupoid semirings C_nG to have pseudo zero divisors, right pseudo zero divisors and left pseudo zero divisors is given.

In view of all these; the following theorem is true.

Theorem 2.3: Let $G = \{Z_m, * (t, s) \mid t + s \equiv 1 \pmod m\}$ be the groupoid. C_n be the chain lattice of order n. C_nG be the groupoid semiring.

$B = \{a_ig \mid a_i \in C_n$ and $g \in G\} \subseteq C_nG$ is a non commutative idempotent groupoid under $\times$.

Proof: Let C_nG be the groupoid semiring. $B = \{a_ig \mid a_i \in C_n, g \in G\} \subseteq C_nG$. Let $x = a_ig \in B$;

$$\begin{aligned} x \times x &= a_ig \times a_ig \\ &= a_i \cap a_i (g * g) \\ &= a_i (tg + sg) \\ &= a_i [(t + s)g] \bmod m \\ &= a_ig = x. \end{aligned}$$

Thus every element in B is an idempotent under $\times$.

Further let $x = a_ig$ and $y = a_jh \in B$.

$$\begin{aligned} x \times y &= a_ig \times a_jh \\ &= a_i \cap a_j (g * h) \\ &= a_kk \text{ (where } a_k = a_i \text{ if } a_i < a_j \\ &\quad a_j \text{ if } a_j < a_i \text{ and } g * h = k \in Z_m). \end{aligned}$$

Clearly $a_kk \in B$. Thus B is a groupoid under $\times$. Since G is a non commutative groupoid B is a idempotent non commutative groupoid of finite order.

Corollary 2.1: Let C_nG and B be as in theorem. $|B| = (n - 1) \times |m| + 1$.
Proof is direct.

Next substructures of these groupoid semirings are analyzed. First subsemirings of groupoid semirings is studied in the following. Presently some examples to this effect are given.

Example 2.8: $G = \{Z_{12}, *, (10, 8)\}$ be the groupoid. $C_{16} = \{0 < a_1 < a_2 < \ldots < a_{14} < 1\}$ be the semiring. $C_{16}G$ be the groupoid semiring.

$P_1 = \{C_{16}H_1 \mid H_1 = \{0, 4, 8\} \subseteq G\} \subseteq C_{16}G$ is the subsemiring of $C_{16}G$. $P_2 = \{C_{16}H_2 \mid H_2 = \{0, 2, 4, 6, 8, 10\} \subseteq G\} \subseteq C_{16}G$ is the subsemiring of $C_{16}G$.

Example 2.9: Let $G = \{Z_{12}, * (2, 10)\}$ be the groupoid. $C_{12} = 0 < a_1 < a_2 < \ldots < a_{10} < 1$ be the chain lattice. $C_{12}G$ be the groupoid semiring.

$B_1 = \{C_{12}B_1 \mid B_1 = \{0, 2, 4, 6, 8, 10\} \subseteq G\} \subseteq C_{12}G$ is a subsemiring of finite order.

Example 2.10: Let $G = \{Z_9, * (5, 3)\}$ be the groupoid. $C_9 = 0 < a_1 < a_2 < \ldots < a_7 < 1$ be the semiring. C_9G be the groupoid semiring.

Let $B_1 = \{C_9H_1 \mid H_1 = \{0, 3, 6\} \subseteq G\} \subseteq C_9G$ and $B_2 = \{C_9 H_2 \mid H_2 = \{1, 2, 4, 5, 7, 8\} \subseteq G\} \subseteq C_9G$ be subsemirings of the groupoid semiring C_9G.

Now the following theorem is proved.

Theorem 2.4: Let $G = \{Z_m, *, (s, t)\}$ be the groupoid. $C_n = \{0 < a_1 < a_2 < \ldots < a_{n-2} < 1\}$ be the semiring. C_nG be the groupoid semiring. C_nG has several subsemirings.

Proof: C_nG be the groupoid semiring.

Let $L_1 = \{0, 1\} \subseteq C_n$ be the sublattice of C_n. $L_1G \subseteq C_nG$; clearly L_1G is a subsemiring of C_nG.

Let $L_2 = \{0, a_i, 1\} \subseteq C_n$ be the sublattice of C_n. $L_2G \subseteq C_nG$; clearly L_2G is a subsemiring of C_nG.

Let $L_3 = \{0, a_i, a_j, 1\} \subseteq C_n$ be the sublattice of C_n. $L_3G \subseteq C_nG$ be the subsemiring of C_nG.

Thus the groupoid semiring contains several subsemirings.

Corollary 2.2: Let $C_2 = \{0, 1\}$ be a two element lattice and G be any groupoid. C_2G is a groupoid semiring of G over the semiring C_2.

Proof: Let C_2G be the groupoid semiring. C_2 has no sublattices. If G has subgroupoids then C_2G has subsemirings.

If G has no subgroupoids then $\{(g_1 + g_2 + \ldots + g_n), 0\}$ is a subsemiring where $G = \{g_1, g_2, \ldots, g_n\}$ and $G * G = G$.

Next topic of study is to find right(left) ideals of the groupoid semirings.

First this situation will be described by some examples.

Example 2.11: Let G be a groupoid given by the following table.

*	a_0	a_1	a_2	a_3
a_0	a_0	a_3	a_2	a_1
a_1	a_2	a_1	a_0	a_3
a_2	a_0	a_3	a_2	a_1
a_3	a_2	a_1	a_0	a_3

$P_1 = \{a_0, a_2\}$ and $P_2 = \{a_1, a_3\}$ are only left ideals of G. Let $C_{20} = \{0 < a_1 < a_2 < \ldots < a_{18} < 1\}$ be the chain lattice. $C_{20}G$ be the groupoid semiring. $C_{20}P_1$ and $C_{20}P_2$ are both left ideals of $C_{20}G$.

Example 2.12: Let G be the groupoid given by the following table. $C_{10} = \{0 < a_1 < a_2 < \ldots < a_8 < 1\}$ be the chain lattice.

*	a_0	a_1	a_2	a_3
a_0	a_0	a_2	a_0	a_2
a_1	a_3	a_1	a_3	a_1
a_2	a_2	a_0	a_2	a_0
a_3	a_1	a_3	a_1	a_3

$P_1 = \{a_0, a_2\}$ and $P_2 = \{a_1, a_3\}$ are right ideals of G. $C_{10}P_1$ and $C_{10}P_2$ are both right ideals of the groupoid semiring $C_{10}G$. Both are not left ideals of $C_{10}G$.

Next examples of ideals in C_nG of a groupoid semiring is given.

Example 2.13: Let $G = \{Z_5, *, (2, 4)\}$ be the groupoid. $C_{11} = \{0 < a_1 < a_2 < \ldots < a_9 < 1\}$ be the chain lattice. $C_{11}G$ be the groupoid semiring.

$P_1 = \{a_r0 + a_i1 + a_j2 + a_k3 + a_l4, 0, a_r, a_i, a_j, a_k, a_l \in C_{11}\}$ is an ideal of $C_{11}G$.

The proof of this follows by analyzing

the table and using the properties of the chain lattice.

*	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

For instance if $x = (a_2 0 + a_4 1 + a_5 2 + a_1 3 + a_3 4) \in P$.

Let $y = a_{10} 4 + a_6 3 \in C_{11}G$.

$$\begin{aligned} x \times y &= (a_2 \cap a_{10})\ (0 * 4) + (a_4 \cap a_{10})\ (1 * 4) + (a_5 \cap a_{10})\ (2 * 4) + (a_1 \cap a_{10})\ (3 * 4) + (a_3 \cap a_{10})\ (4 * 4) + (a_2 \cap a_6)\ (0 * 3) + (a_4 \cap a_6)\ (1 * 3) + (a_5 \cap a_6)\ (2 * 3) + (a_1 \cap a_6)\ (3 * 3) + (a_3 \cap a_6)\ (4 * 3) \\ &= a_2 1 + a_4 3 + a_5 0 + a_1 2 + a_3 4 + a_2 2 + a_4 4 + a_5 1 + a_1 3 + a_3 0 \\ &= (a_2 \cup a_5)1 + (a_4 \cup a_1)3 + (a_5 \cup a_3)0 + (a_1 \cup a_2)2 + (a_3 \cup a_4)4 \\ &= a_5 1 + a_4 3 + a_5 0 + a_2 2 + a_4 4 \in P. \end{aligned}$$

Hence the claim.

Now on similar lines $y \times x \in P$. Thus P is an ideal of $C_{11}G$.

Example 2.14: Let $G = \{Z_{10}, *, (3, 7)\}$ be the groupoid. $C_{12} = \{0 < a_1 < a_2 < \ldots < a_{10} < 1\}$ be the chain lattice $C_{12}G$ be the groupoid semiring.

$C_{12}G$ has no left ideals or right ideals.

$P = \{0, a_{i_1} 0 + a_{i_2} 1 + a_{i_3} 2 + \ldots + a_{i_9} 9 \mid a_{i_j} \in C_{11} \setminus \{0\};\ \} \subseteq C_{12}G$ is an ideal of $C_{12}G$.

To find right ideals which are not left ideals and left ideals which are not right ideals is carried out first by example then by theorem.

Example 2.15: Let $G = \{Z_{16}, (2, 2), *\}$ be the groupoid. C_9 be the chain lattice. C_9G be the groupoid semiring. C_9G has ideals given by $P = \{0, 2, 4, 6, 8, 10, 12, 14\} \subseteq G$ is such that C_9P is an ideal of C_9G.

Example 2.16: Let $G = \{Z_{24}, *, (3, 3)\}$ be the groupoid. C_{12} be the chain lattice. $C_{12}G$ be the groupoid semiring. $C_{12}G$ has ideals as $P = \{0, 3, 6, 9, 12, 15, 18, 21\} \subseteq G$ is an ideal of G and such that $C_{12}P$ is an ideal of $C_{12}G$.

In view of all these the following result is proved.

Theorem 2.5: Let $G = \{Z_n, (t, t); *\}$ be the groupoid. C_m be the chain lattice. C_mG be the groupoid semiring. C_mG has ideals of the form C_mP if t/n.

Proof: If t/n then G has a subgroupoid generated by P (say P). Then $C_mP \subseteq C_mG$ is an ideal of C_mG.

Hence the claim.

Corollary 2.3: If $t \nmid n$ then G has no subgroupoid. Thus C_mG has no ideals of the form C_mP; $P \subseteq G$.

Corollary 2.4: If n is a prime; G has no subgroupoid. So C_mG has no ideals of the form C_mP; $P \subseteq G$.

The corollaries are clear from the fact

$G = \{Z_n, (t, t), *\}$ has no subgroupoids if $(n, t) = 1$. This always occurs when n is a prime number. Thus C_nG has no ideals of the form C_mP; $P \subseteq G$; P an ideal of G.

Example 2.17: Let $G = \{Z_{15}, (0, 7), *\}$ be the groupoid. C_{15} be the chain lattice; $0 < a_1 < a_2 < \ldots < a_{13} < 1$.

C_mG be the groupoid semiring. As G has no subgroupoids, so C_mG has no ideals of the form C_mP; P an ideal of G.

Example 2.18: Let $G = \{Z_{15}, (0, 5), *\}$ be the groupoid. $C_{15} = 0 < a_1 < \ldots < a_{13} < 1$ be the chain lattice and $C_{15}G$ be the groupoid semiring.

$P = \{0, 5, 10\} \subseteq G$ be the ideal of G. C_mP is an ideal of C_mG.

If $(0, 5)$ is replaced by $(3, 0)$ then $P = \{0, 3, 6, 9, 12\} \subseteq G$ is an ideal of G. C_mP is an ideal of C_mG.

If $(2, 0)$ is taken then G has no subgroupoids and C_mG has no ideals of the form C_mP.

In view of all these the following result is proved.

Theorem 2.6: Let $G = \{Z_m, *, (t, 0)\}$ be the groupoid. C_n be the chain lattice. C_nG has an ideal of the form C_nP if and only if t / m.

Proof: P is an ideal of G if and only if t / m.

Hence C_nP is an ideal of C_nG if and only if t / m.

Corollary 2.5: Let $G = \{Z_p, *, (t, 0)\}$, p a prime. C_n be the chain lattice. C_nG has no ideals of the form C_nP.

Proof: Follows from the fact for no $t \in Z_p \setminus \{1\}$ is such that t / p.

3. Conclusions

In this paper for the first time the notion of groupoid semirings using semirings as lattices is constructed. Several important properties in this direction is developed. This is a class of finite non associative semirings.

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