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Quasi-Pure W-Projective Modules and Dimension

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Abstract

In this paper, we generalize the notion of projective, injective and flat modules and dimension. Hence, we introduce and study the notion of Quasi-pure W-projective modules and dimension.

Key words: classical projective dimension, quasi-Frobenius ring, FP-injective, pure projective, cotorsion dimension.

Introduction

Throughout this paper, all rings are associative and all modules if not specified otherwise are left and unitary. Let R is a ring and M be an R-module. As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote respectively, the classical projective dimension, injective dimension and flat dimension of M. We use also $\mathrm{qpwpd}_R(M)$, gldim(R) and Wdim(R) to denote respectively, the quasi-pure W-projective dimension of M, classical global and Quasi W-dimension of R. The character module $\mathrm{Hom}_Z\left(M, \frac{Q}{z}\right)$ is denoted by M^1.

Recall that a ring R is called left perfect, if every flat module is projective. Example of Perfect rings includes quasi-Frobenius rings that are right or left self injective rings and right or left Artinian. It is shown that a ring is quasi - Frobenius if and only if every projective module is injective, if and only if every injective module is projective. We introduce and study a new generalization of projective and injective modules any dimension[3,5].

The relation between the quasi-pure W-projective dimension and other dimensions are discussed.

Definition 1.1: A module is said to be pure-projective if it is projective with respect to pure-exact sequences- Warfield[2] showed that a module is pure-projective if and only if it is a direct summand of a direct sum of finitely presented modules.

Definition 1.2: For an R-module M, the quasi-pure W-projective dimension of M,

$qpwpd_R(M)$, is defined to be the smallest integer n≥0 such that $Ext_R^{n+1}(M, M) = 0$ for all flat modules M. If no such integer exists, set $qpwpd_R(M) = \infty$. If $qpwpd_R(M) = 0$ then M will be called a quasi-pure W-projective module.

Example 1.3: Consider the local quasi-Frobenius ring $R=\frac{m[z]}{z^2}$ where m is a field and denoted by $\bar{z}$ the residue class in R of z. Then $\bar{z}$ is a quasi-pure W-projective R-module which is not projective[8].

Proof: Since R is quasi-Frobenius, every projective (and every flat since R is Perfect) modules M is injective. Then $Ext_R^i(\bar{z}, M)=0$ for all i > 0. Thus $\bar{z}$ is a quasi-pure W-projective R-module. Now, if we suppose that $\bar{z}$ is projective. Then, it must be free since R is local, a contradiction since $\bar{z}^2= 0$. So, we conclude that $\bar{z}$ is not projective, as desired.

The authors[6] defined and studied a refinement of flat modules which they called the IF modules. Recall that an R-module M is said IF Module if $Tor_r^i(I, M) = 0$ for all right injective R-module and all i> 0.

Proposition 1.4: Let R be a right coherent ring. Then every quasi-pure W-projective R-module is an IF R-module.

Proof: Let M be a quasi-pure W-projective R-module. Let E be an injective right R-module. Then $\bar{E}$ is flat. Then $Ext_R^i(M, \bar{E})=0$ for all i>o. While $Ext_R^i(M, \bar{E})= \overline{\left(Tor_R^i(\bar{E}, M)\right)}$. Hence $Tor_r^i(E, M) = 0$. Thus M is an IF-module.

Proposition 1.5: Let M be a quasi-pure W-projective R-module, then

(i) $Ext_R^i(M, M^*) = 0$ for all i>0 and all M^* with finite flat dimension.

(ii) Either M is Projective or $fd_R(M) = \infty$.

Proof: (i) Since $Ext_R^i(M, M^*) = 0$ for all flat modulesand M^* all i>0, the proof is immediate by dimension shifting.

(ii) Suppose that $fd_R(M)<\infty$ and pick a short exact Sequence $0 \to M^* \to P \to M \to 0$ where P is projective. Clearly $fd_R(M^*) < \infty$ then $Ext_R^i(M, M^*) = 0$. Thus the short exact sequence splits and so M is isomorphic to a direct summand of P and then projective.

Corollary 1.6: A module M is quasi-pure Projective if and only if it is flat and quasi-pure W-projective.

Proof: Let M be an R-module. The cotorsion dimension of M, $cd_R(M)$ is smallest integer n such that $Ext_R^{n+1}(\bar{M}, M)= 0$ for all flat modules $\bar{M}$. The left cotorsion dimension of the R, cot.D(R) is the supremum of cotorsion dimension of R module. It is shown in [7, corollary 7.2.6] that l.cot.D(R) $= \mathrm{Sup}\{qpd_R(\overline{M}) / \overline{M}\text{ - flat}\}$.

Proposition 1.7: Let M be an R-module and consider the following conditions.

(i) M is a quasi-pure W-projective module.

(ii) $\mathrm{Ext}^i_R(M, P) = 0$ for all i>0 and the projective modules P.

(iii) $\mathrm{Ext}^i_R(M, P) = 0$ for all $i > 0$ and all module P with finite projective dimension.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)

All statements are equivalent if M is finitely presented or if l.cot.D (R) $< \infty$.

Proof: (i) $\Rightarrow$ (ii) it is trivial

(ii) $\Leftrightarrow$ (iii) Result by dimension shifting.

Let $\overline{M}$ be a flat module. By Lazard's theorem [4, section 1 N. 6 theorem 1], there is a direct system $(L_i)_{i\in I}$ of finitely generated free R-modules such that $\varinjlim L_i \cong \overline{M}$.

If M is finitely presented, from [4, Exercise 3, P-187]. We have $\mathrm{Ext}^i_R(M, \overline{M}) \cong \varinjlim \mathrm{Ext}^i_R(M, L_i)$. Thus in this case the implication (iii)$\Rightarrow$(i) holds.

If l.cot.D(R)$<\infty$ then $qpd_R(\overline{M}) < \infty$. Hence in this case also the implication (iii)$\Rightarrow$(i) holds[11].

Proposition 1.8: The following statements are equivalent.

(i) R is left perfect

(ii) Every flat module is quasi-pure W-projective.

In particular if the class of all quasi-pure W-projective modules is closed under direct limits, then R is left perfect.

Proof: If R is left perfect it is clear that every flat module is quasi-pure W-projective. As to the converse, let $\overline{M}$ be a flat module. By (i) it is quasi-pure W-projective and so projective by proposition 1.5. Then R is left perfect. If the class of all quasi-pure W-projective module is closed under direct limits, then any direct limit of projective modules is both flat and quasi-pure W-projective (since every projective module is both flat and quasi-pure W-projective). Then by corollary 1.6 every direct limit of projective modules is projective. Thus R is left perfect.

Proposition 1.9: The following are equivalent.

(i) Every R-module is quasi-pure W-projective.
(ii) R is quasi- Frobenius.

Proof: This follows from the fact that a ring is quasi-Frobenius if and only if every projective module is injective and that is quasi-Frobenius rings are perfect.

A left (right) R-module M is said FP-injective if $\mathrm{Ext}^i_R(M, M) = 0$ for every finitely presented left (right) R-module M.

A ring R is said to be FC if it is left and right coherent and left right self FP-injective.

Proposition 1.10: The following are equivalent.

(i) R is FC.

(ii) Every finitely presented (left and right) module is quasi-pure W-projective.

Proof: Let M be a finitely presented right (or left) module and $\overline{M}$ be a flat right (or left) module. Then $\overline{M}$ is FP-injective by [10, lemma 4.1]. So, $\mathrm{Ext}^i_R(M, \overline{M}) = 0$ for all i>0 .Thus M is quasi-pure W-projective. As to converse for any finitely presented right of left module M we have $\mathrm{Ext}^i_R(M, R) = 0$ for all i>0 by (ii). Thus R is self right and left FP-injective.

Proposition 1.11: For any R-module M and any Positive integer n, the following assertions are equivalent.

(i) $\mathrm{qpwpd}_R(M) \le n$

(ii) $\mathrm{Ext}^i_R(M, \overline{M}) = 0$ for all i>n and all R-module $\overline{M}$ with f inite flat dimension.

(iii) If $0 \to G_n \to G_{n-1} \to \ldots\ldots \to G_0 \to M \to 0$ is an exact sequence of modules with $G_0 \ldots\ldots G_{n-1}$ are quasi-pure W-projective modules, Then G_n is a quasi W-projective module.

Proof: (i)$\Leftrightarrow$ The proof of this equivalence is standard homological algebra.

(ii)$\Rightarrow$(i) Obvious.

(i)$\Rightarrow$(ii) Set $P = \mathrm{qpwpd}_R(M)$. By induction on $m = \mathrm{fd}_R(\overline{M})$, we prove that $\mathrm{Ext}^i_R(M, \overline{M}) = 0$ for all i>p. The induction start is given by (i). If m>0 pick the short exact sequence $0 \to \overline{M}' \to P \to \overline{M} \to 0$ where P is a projective module. Clearly $\mathrm{fd}_R(\overline{M}') = m-1$. Thus $\mathrm{Ext}^i_R(M, \overline{M}') = 0$ for all i>n. From the long exact sequence

$$\to \mathrm{Ext}^i_R(M, P) \to \mathrm{Ext}^i_R(M, \overline{M}) \to \mathrm{Ext}^{i+1}_R(M, \overline{M}') \to \ldots\ldots$$

It is clear that $\mathrm{Ext}^i_R(M, \overline{M}) = 0$ for all i> n.

Proposition 1.12: For any R-module M, $\mathrm{qpwpd}_R(M) \le \mathrm{pd}_R(M)$, with equality if $\mathrm{fd}_R(M)$ is finite.

Proof: The first inequality follows from the fact that every projective module is quasi-pure W-projective. Now, set $\mathrm{qpwpd}_R(M) = n < \infty$ and consider an n-step projective resolution of M as follows.

$$0 \to M' \to P_{n-1} \to \ldots\ldots\ldots\ldots \to P_1 \to P_0 \to M \to 0$$

Where all P_i are projective. Clearly, M' is quasi-pure W-projective. If $\mathrm{fd}_R(M) < \infty$ then

$fd_R(M') < \infty$ and then it is projective by proposition 1.5. Hence $pd_R(M) \leq n$, and so the equality holds.

[2]. Quasi-pure W-Projective dimension of rings:

Definition 2.1: The left quasi-pure W-projective dimension of a ring R, l. qpwpd(R) is defined by setting l. qpwpd(R) = Sup{ qpwpd(M) / M is a (left) R-module}.

Theorem 2.2: Let R be a ring and n be a positive integer. The following are equivalent.

(i) l. qpwpd(R) $\leq$ n.

(ii) qpwpd$(R/I) \leq$n, for every (left) ideal I of R.

(iii) $id_R(\overline{M}) \leq n$, for all flat modules $\overline{M}$.

(iv) $id_R(p) \leq n$, for all Projective modules P.

Proof: (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious.

(ii)$\Rightarrow$(iii) Let $\overline{M}$ be a flat module. Since $qpwpd_R(R/I) \leq n$ for every ideal I of R, we have $Ext^i_R(R/I, \overline{M}) = 0$ for all i>n. Thus using the Baer Criterion [9, lemma 9.11], $id_R(\overline{M}) \leq n$.

(iv)$\Rightarrow$(i) Let M be an arbitrary module. Since $id_R(p) \leq n$ for each Projective module P, we have $Ext^i_R(M, P) = 0$ for all i> n and Projective module P.

By dimension shifting we get that $Ext^i_R(M, P) = 0$ for all i>n and all module P with finite projective dimension. By [7, theorem 7.2.5] l.Cot.D(R)≤Sup{id_R(M)/p projective}≤n. Thus given a flat module $\overline{M}$, we have $qppd_R(\overline{M}) < \infty$. Hence, $Ext^i_R(M, \overline{M}) = 0$ for all i>n. Consequently, $qpwpd_R(M) \leq n$.

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