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Hydromagnetic Radiative Convection Flow in a Vertical Channel With Temperature Dependent Viscosity and Viscous Dissipation and Heat Generation/Absorption

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Abstract

Fully developed radiative convection flow of an electrically conducting, incompressible viscous liquid in a long vertical channel consisting of two parallel impermeable plates is investigated. The flow exists under constant pressure gradient, temperature dependent viscosity, viscous dissipation and in the presence of transversely applied uniform magnetic field. To obtain the solutions of the coupled non-linear system of equations, the regular perturbation technique is used. The effects of different physical parameters on velocity field, temperature distribution, skin-friction and rate of heat transfer are numerically evaluated, presented graphically and discussed.

Key words : Radiative convection, impermeable walls, vertical channel, viscous dissipation, heat generation/absorption.

Nomenclature

C_p : specific heat at constant pressure,
Ec : Eckert number,
Gr : Grashof number,
g : acceleration due to gravity,
h : half width of the channel,
K_T : thermal conductivity,
M : magnetic parameter,
N : radiation parameter,
Nu : rate of heat transfer (Nusselt number),

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P : non-dimensional pressure,

P' : $\left(=-\frac{1}{\rho}\frac{\partial p'}{\partial x'}\right)$ constant pressure gradient,

Pr : Prandtl number,

Q : heat source/sink parameter,

q'_r : radiative heat flux,

T : non-dimensional temperature of the fluid,

T' : dimensional temperature of the fluid,

T'_1, T'_2 : temperatures of the left and right channel walls,

u : non-dimensional velocity of the fluid along the x-axis,

u' : velocity of the fluid along the x'-axis,

u_m : mean velocity in the channel,

x, y : non-dimensional coordinates,

x', y' : dimensional coordinates,

Greek symbols

β' : small positive constant,

β : viscosity parameter,

β_1 : coefficient of thermal expansion,

μ_0 : constant viscosity when $\beta' = 0$,

μ' : variable viscosity of the fluid,

ρ : density of the fluid,

τ : skin-friction at the walls.

I. Introduction

The viscosity of many fluids varies with temperature. Therefore, the results obtained from flow of fluids with constant viscosity are not applicable for the fluids with temperature dependent viscosity, particularly at high temperature differences. Investigated steady non-Newtonian free convection radiative heat transfer past a vertical porous plate in the presence of uniform magnetic field[1]. Studied convective heat transfer along a vertical surface in a saturated porous medium and observed the effect of variable viscosity[2]. Investigated variable viscosity effects and temperature dependent viscosity effects on convective flow under different physical situations[3,4,5,6,7,8]. Considered temperature dependent viscosity between two parallel plates and couette flow of a dusty fluid with temperature dependent thermal conductivity respectively[9,10]. Analysed radiation effects on mixed convective flow past an moving vertical plate for high temperature differences in the presence of uniform magnetic field[11]. Recently, discussed mixed convection flow with temperature dependent viscosity past a vertical flat plate[12]. More recently, have discussed convective flow problems for different physical parameters under various physical situations with constant[13,14].

Studied fully developed flow with temperature dependent viscosity in a horizontal channel governed by constant pressure gradient and viscous dissipation effect considering that the variable viscosity decreases exponentially with temperature and follows Arrhenius model[15]. The channel is considered horizontal with isothermal walls. Studied the model for flow in the horizontal channel embedded in porous medium[7]. Now it is proposed to investigate hydromagnetic flow of an electrically conducting, incompressible, viscous liquid through a vertical channel in the presence of thermal radiation and absorption type heat source. The viscosity and magnetic field are considered to be variable and depend on temperature.

II. Formulation of the Problem :

Fully developed radiative convection flow of an electrically conducting, incompressible viscous liquid through a long channel consisting of two parallel isothermal flat walls. In two-dimensional cartesian coordinate system (x' , y'), the x' -axis is chosen in the middle of the channel walls and y' -axis is taken normal to it. A uniform magnetic field is applied along the y' -axis, *i.e.*, perpendicular to the flow. The distance between the channel walls is $2h$ and T_1' , T_2' are temperatures of the channel walls. The physical model and coordinate system is shown in Fig. 1.

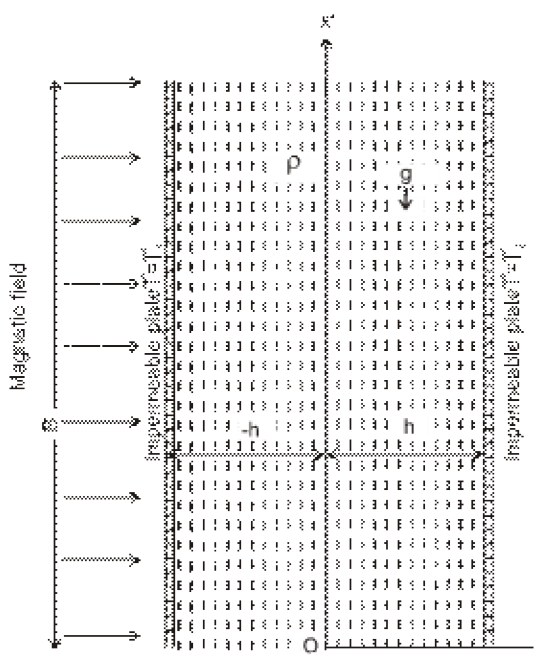

Fig. 1: Physical model and coordinate system.

The flow is governed by constant pressure gradient $-\frac{1}{P}\frac{\partial p}{\partial x} = -P'$ and usual Boussinesq approximation is taken into account. The fluid viscosity is high and is assumed variable, which decreases exponentially with temperature[15]. Therefore, $\mu' = \mu_0 \exp\{-\beta'(T_2' - T_1')\}$. The variable magnetic field is assumed as $B = B_0 \exp\left\{-\frac{\beta'}{2}(T_2' - T_1')\right\}$. Viscous dissipation term and Rosseland approximation for the radiative heat flux reactor is considered in the energy equation[16]. The walls of the channel are long enough, as such all the variables are the functions of y' only. The magnetic Reynolds number is assumed small so that induced magnetic field is negligible.

Under the above restrictions, the flow is governed by the following equations:

$$\frac{d}{dy'}\left(\frac{\mu'}{\rho}\frac{du'}{dy'}\right) + g\beta_1 (T' - T_1') - \frac{\sigma}{\rho}B^2 u' - P' = 0 \tag{1}$$

$$\frac{K_T}{\rho C_p}\frac{d^2T'}{dy'^2} - \frac{1}{\rho C_p}\frac{dq_r'}{dy'} + \frac{\mu'}{\rho C_p}\left(\frac{du'}{dy'}\right)^2 - \frac{Q'}{\rho C_p}(T' - T_1') = 0 \tag{2}$$

For an optically thick fluid, in addition to emission, there is also self absorption[11] and the absorption coefficient depends on wave length[1]. Hence, we can use the Rosseland approximation for the radiative heat flux vector.

$$q_r' = -\frac{4\sigma^*}{3k^*}\frac{dT'^4}{dy'} . \tag{3}$$

σ^* is the Stefan-Boltzmann constant and k^* is the absorption coefficient. Assuming small temperature differences within the flow T'^4, may be expanded in a Taylor's series about the temperature T_1' as follows:

$$T'^4 = T_1'^4 + 4T_1'^3(T'-T_1') + 6T_1'^2(T'-T_1')^2 + \ldots\ldots\ldots\ldots$$

Neglecting higher order terms in the above series beyond the first degree in $(T'-T_1')$, we get:

$$T'^4 \cong 4T_1'^3 T' - 3T_1'^4 \qquad (4)$$

The boundary conditions relevant to the problem are:

$$u' = 0, \quad T' = T_1' \quad \text{at} \quad y' = -h,$$

$$u' = 0, \quad T' = T_2' \quad \text{at} \quad y' = h. \qquad (5)$$

The symbols are defined in the nomenclature.

We introduce the following non-dimensional quantities and parameters:

$$u = \frac{u'}{u_m}, \quad y = \frac{y'}{h}, \quad \mu = \frac{\mu'}{\mu_0}, \quad T = \frac{T' - T_1'}{T_2' - T_1'}.$$

$$Pr = \frac{\mu_0 C_p}{K_T}, \quad P = \frac{\rho P' h^2}{\mu_0 u_m}, \quad N = \frac{16\sigma^* T_1'^3}{3k^* K_T},$$

$$Gr = \frac{g\rho\beta_1(T_2' - T_1')h^2}{\mu_0 u_m}, \quad Ec = \frac{u_m^2}{C_p(T_2' - T_1')}$$

$$M = B_0 h\sqrt{\frac{\sigma}{\mu_0}} \text{ and } Q = \frac{Q' h^2}{\mu_0 C_p}.$$

Substituting μ', B, q_r' and above mentioned non-dimensional quantities and physical parameters, the equations (1) and (2) transform to:

$$\frac{d}{dy}\left(\exp(-\beta T)\frac{du}{dy}\right) + GrT - M^2\exp(-\beta T)u = P. \qquad (6)$$

$$\frac{d^2T}{dy^2} + \frac{Ec\,Pr}{1+N}\exp(-\beta T)\left(\frac{du}{dy}\right)^2 - \alpha^2 T = 0, \qquad (7)$$

where $\beta = \beta'\left(T_2' - T_1'\right)$ and $\alpha = \sqrt{\frac{PrQ}{1+N}}$.

The boundary conditions (5) reduce to :

$$u=0, \quad T=0 \quad \text{at} \quad y = -1$$

$$u=0, \quad T=1 \quad \text{at} \quad y = 1. \qquad (8)$$

Now, we proceed to obtain the solution of equations (6) and (7) under the boundary conditions (8).

Solution of the Problem :

Since the exact solution of these coupled non-linear equations is not possible, we expand u and T in the powers Eckert number, Ec (for incompressible liquids the Eckert number Ec is always <<1) as follows:

$$u = u_0 + Ecu_1 + Ec^2u_2 + \ldots\ldots\ldots \text{ and}$$

$$T = T_0 + EcT_1 + Ec^2T_2 + \ldots\ldots\ldots. \qquad (9)$$

Introducing (9) in equations (6) and (7) and comparing the coefficient of Ec^0 and Ec^1, we obtain:

$$\frac{d^2u_0}{dy^2}-\beta\frac{dT_0}{dy}\frac{du_0}{dy}-M^2u_0=(P-GrT_0)\exp(\beta T_0). \quad (10)$$

$$\frac{d^2u_1}{dy^2}-\beta\frac{dT_0}{dy}\frac{du_1}{dy}-M^2u_1=\beta\frac{dT_1}{dy}\frac{du_0}{dy}$$
$$+\beta(P-GrT_0)T_1\exp(\beta T_0)-GrT_1\exp(\beta T_0). \quad (11)$$

$$\frac{d^2T_0}{dy^2}-\alpha^2T_0=0. \quad (12)$$

$$\frac{d^2T_1}{dy^2}-\alpha^2T_1=\frac{Pr}{1+N}\exp(\beta T_0)\left(\frac{du_0}{dy}\right)^2. \quad (13)$$

The boundary conditions (8) are transformed as follows:

$$u_0=0,\ u_1=0,\ T_0=0,\ T_1=0 \text{ at } y=-1.$$
$$u_0=0,\ u_1=0,\ T_0=1,\ T_1=0 \text{ at } y=1. \quad (14)$$

The solution of equation (12) is obtained in terms of exponential powers of α. In the expanded form, we retain upto first term in α. Thereafter, the solution of this couplied equations (10), (12) and (13) satisfying the boundary conditions (14) are obtained as follows:

$$T_0=K_1+K_2y. \quad (15)$$

$$u_0=C_1\exp(m_1)y+C_2\exp(m_2)y+K_3\exp(\beta K_2)y$$
$$+K_4y\exp(\beta K_2)y. \quad (16)$$

$$T_1=C_3\exp(\alpha)y+C_4\exp(-\alpha)y+K_5\exp(2m_1-K_2)y$$
$$+K_6\exp(2m_2-K_2)y+K_7\exp(2\beta-1)+K_8\exp(m_1+m_2-K_2)$$
$$+K_9\exp(m_1+\beta K_2-K_2)y+K_{10}\exp(m_2+\beta K_2-K_2)y$$
$$+K_{11}y^2\exp(2\beta-1)K_2y+K_{12}y\exp(2\beta-1)K_2y+K_{13}y$$
$$\exp(m_1+\beta K_2-K_2)y+K_{14}y\exp(m_2+\beta K_2-K_2)y. \quad (17)$$

$$u_1=C_5\exp(m_1y)+C_6\exp(m_2y)+K_{17}\exp(\alpha+m_1)y$$
$$+K_{18}\exp[-(\alpha-m_1)y]+K_{19}\exp(\alpha+m_2)y$$
$$+K_{20}\exp[-(\alpha-m_2)]y+K_{21}\exp(\alpha+\beta K_2)y$$
$$+K_{22}\exp[-(\alpha-\beta K_2)]y+K_{23}\exp(3m_1-K_2)y$$
$$+K_{24}\exp(3m_2-K_2)y+K_{25}\exp(3\beta-1)K_2y$$
$$+K_{26}\exp(m_1+2m_2-K_2)y+K_{27}\exp(2m_1+m_2-K_2)y$$
$$+K_{28}\exp(m_1+2\beta K_2-K_2)y+K_{29}\exp(m_2+2\beta K_2-K_2)y$$
$$+K_{30}\exp(2m_1+\beta K_2-K_2)y+K_{31}\exp(2m_2+\beta K_2-K_2)y$$
$$+K_{32}\exp(m_1+m_2+\beta K_2-K_2)y+K_{33}y\exp(m_1+2\beta K_2-K_2)y$$
$$+K_{34}y\exp(m_2+2\beta K_2-K_2)y+K_{35}y\exp(2m_1+\beta K_2-K_2)y$$
$$+K_{36}y\exp(2m_2+\beta K_2-K_2)y+K_{37}y\exp(m_1+m_2+\beta K_2-K_2)y$$
$$+K_{38}y\exp(3\beta-1)K_2y+K_{39}y\exp(\alpha+\beta K_2)y$$
$$+K_{40}y\exp[-(\alpha-\beta K_2)]y+K_{41}y^2\exp(m_1+2\beta K_2-K_2)y$$
$$+K_{42}y^2\exp(m_2+2\beta K_2-K_2)y+K_{43}y^2\exp[-(3\beta-1)]K_2y$$
$$+K_{44}y^3\exp[-(3\beta-1)]K_2y+K_{53}y\exp(\alpha+K_2)y$$
$$+K_{54}y\exp[-(\alpha-K_2)]y+K_{55}y\exp(2m_1y)+K_{56}y\exp(2m_2y)$$
$$+K_{58}y\exp(m_1+m_2)y+K_{62}y^2\exp(m_1+2\beta K_2)y$$
$$+K_{63}y^2\exp(m_2+2\beta K_2)y+K_{64}y^3\exp(2\beta K_2y)$$
$$+K_{74}\exp(\alpha+K_2)y+K_{75}\exp[-(\alpha-K_2)]y+K_{76}\exp(2m_1y)$$
$$+K_{77}\exp(2m_2y)+K_{78}\exp(2\beta K_2y)+K_{79}\exp(m_1+m_2)y$$
$$+K_{80}\exp(m_1+\beta K_2)y+K_{81}\exp(m_2+\beta K_2)y$$
$$+K_{82}\exp(2\beta K_2y)+K_{83}y\exp(m_1+\beta K_2)y$$

$$+K_{84}y\exp(m_2+\beta K_2)y+K_{85}y^2\exp(2\beta K_2 y). \quad (18)$$

IV. Skin-Friction and Rate of Heat Transfer:

Knowing the velocity field and temperature distribution, we now calculate the skin-friction (τ) and the rate of heat transfer in terms of Nusselt number (*Nu*).

The skin-friction (τ) and the rate of heat transfer (*Nu*) at the channel walls ($y=\pm1$) are:

$$\tau=\left.\frac{du}{dy}\right|_{y=\pm1}=\left.\frac{du_0}{dy}\right|_{y=\pm1}+Ec\left.\frac{du_1}{dy}\right|_{y=\pm1} \quad (19)$$

and $$Nu=\left.\frac{dT}{dy}\right|_{y=\pm1}=\left.\frac{dT_0}{dy}\right|_{y=\pm1}+Ec\left.\frac{dT_1}{dy}\right|_{y=\pm1}. \quad (20)$$

1. Particular case :

If heat source/sink parameter, Q=0, Grashof number Qr=0 and thermal radiation parameter, N=0; the results of the present study are exactly the same to those obtained by Kankane and Gokhale[22] exact notations.

2. Result and Conclusions

In order to get insight of the problem, the effects of the physical parameters calculated, presented graphically and discussed. The values of Grashof number are chosen for cooling of the channel walls (*Gr*>0) due to practical applications in nuclear technology and energy system. To observe the effect of the parameter, the remaining parameters are kept fixed.

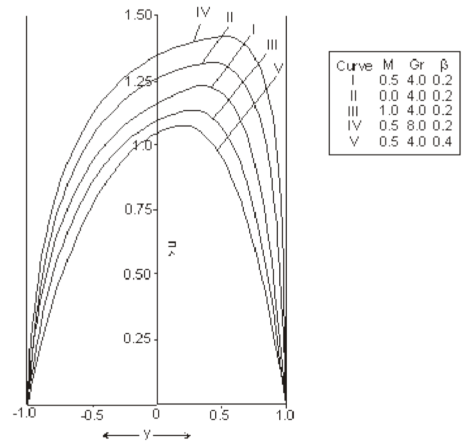

Fig. 2: Velocity profiles for different values of M, Gr and β.
(P=1.0, Ec=0.1, Pr=0.71, Q= -5.0 and N= 4.0).

Fig. 2 is illustrated to demonstrates the velocity field versus non-dimensional y for different numerical values M, Gr and β at fixed values P=1.0, Ec=0.1, Pr=0.71, Q=–5.0 and N=4. It is observed that an increase in M or β decreases the velocity, whereas an increase in Gr increases the velocity significantly.

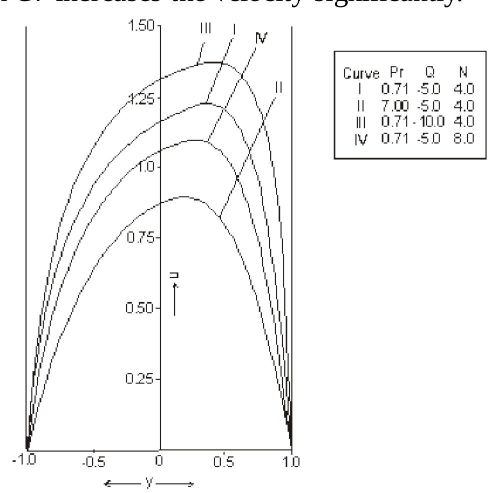

Fig. 3: Velocity profiles for different values of Pr, Q and N.
(P=1.0, Ec=0.1, M=0.5, Gr=4.0 and β.= 0.2).

Fig. 3 shows that effects of *Pr*, *Q* and *N*. It is noted that increase in *Q* with negative sign increases the velocity because the negative value *Q* implies heat source. Increased numerical values of *Pr* or *N* decrease the velocity.

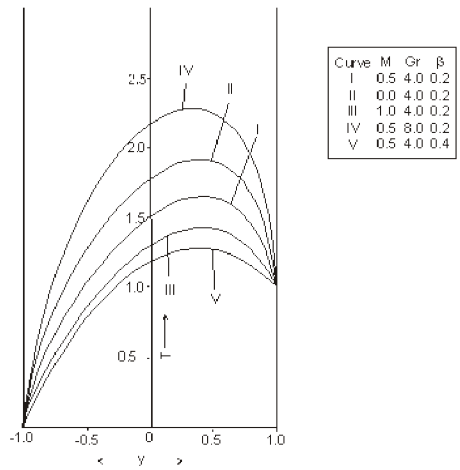

Fig. 4: Temperature profiles for different values of M, Gr and β .
(P = 1.0, Ec = 0.1, Pr = 0.71, Q = -5.0 and β.= 4.0).

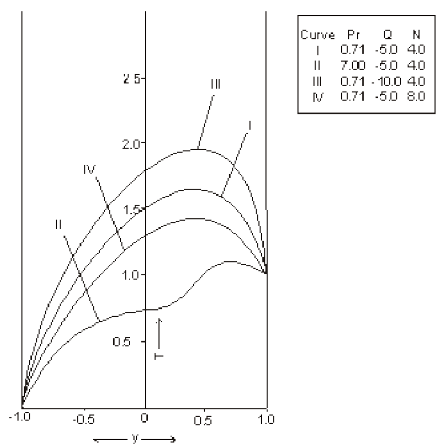

Fig. 5: Temperature profiles for different values of Pr, Q and N.
(P = 1.0, Ec = 0.1, M = 0.5, Gr = 4.0 and β.= 0.2).

Fig. 4 and Fig.5 illustrate the temperature distribution against *y*. The effects of *M*, *Gr* and β are presented in Fig. 4, whereas effects of *Pr*, *Q* and *N* are shown in Fig. 5. We note that increase in *M*, β, *N* or *Pr* decrease the temperature, whereas increase in *Gr* or *Q* results in increase in temperature.

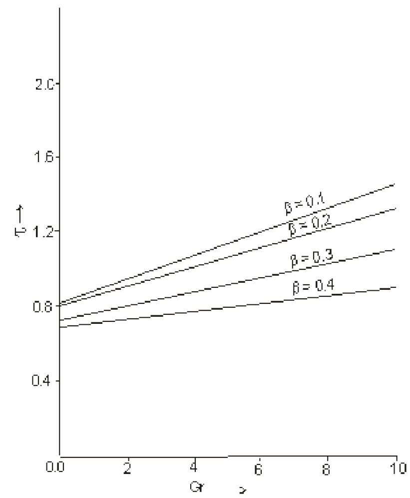

Fig. 6: Skin-friction (τ) with respect to Gr for different values of β.

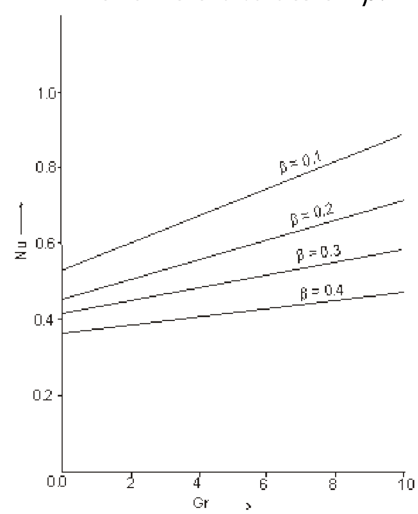

Fig. 7: Rate of heat transfer (Nu) with respect to Gr for different values of β.

Fig. 6 and Fig.7 show the skin-friction versus *Gr* and rate of heat transfer (*Nu*) against *Gr* respectively. It is seen that as β increases the skin-friction (τ) and the Nusselt number decrease.

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APPENDIX

$$K_1 = \frac{1}{\exp(\alpha) + \exp(-\alpha)},$$

$$K_2 = \frac{\alpha}{\exp(\alpha) - \exp(-\alpha)},$$

$$K_3 = \frac{Gr \exp(\beta K_1)}{M^4}\left(K_1 M^2 + \beta K_2^2\right) - \frac{P \exp(\beta K_1)}{M^2} K_4$$

$$= \frac{Gr K_2 \exp(\beta K_1)}{M^2}, \quad K_5 = -\frac{Pr \exp(-K_1) C_1^2}{(1+N)\left[(2m_1 - K_2)^2 - \alpha^2\right]}$$

$$C_1 = \frac{(K_3 - K_4)\exp(m_2 - \beta K_2) - (K_3 + K_4)\exp[-(m_2 - \beta K_2)]}{\exp(m_1 - m_2) - \exp[-(m_1 - m_2)]}$$

$$C_2 = \frac{(K_4 - K_3)\exp(m_1 - \beta K_2) + (K_3 + K_4)\exp[-(m_1 - \beta K_2)]}{\exp(m_1 - m_2) - \exp[-(m_1 - m_2)]}$$

$$K_6 = -\frac{Pr \exp(-K_1) C_2^2}{(1+N)\left[(2m_2 - K_2)^2 - \alpha^2\right]}$$

$$K_7 = \frac{4Pr \exp(-K_1) K_4 K_2 K_3 (2\beta - 1)}{(1+N)\left[(2\beta - 1)^2 - \alpha^2\right]^3}$$

$$-\frac{2Pr \exp(-K_1) K_4^2 \left\{3(2\beta-1)^2 + \alpha^2\right\}}{(1+N)\left[(2\beta-1)^2 - \alpha^2\right]^3} - \frac{Pr \exp(-K_1) K_3^2}{(1+N)\left[(2\beta-1)^2 - \alpha^2\right]},$$

$$K_8 = -\frac{2Pr \exp(-K_1) C_1 C_2}{(1+N)\left[(m_1 + m_2 - K_2)^2 - \alpha^2\right]},$$

$$K_9 = -\frac{2Pr \exp(-K_1) C_1}{(1+N)\left[(m_1 + \beta K_2 - K_2)^2 - \alpha^2\right]}\left[K_3 - \frac{2K_4(m_1 + \beta K_2 - K_2)}{(m_1 + \beta K_2 - K_2)^2 - \alpha^2}\right]$$

$$K_{10} = -\frac{2Pr \exp(-K_1) C_2}{(1+N)\left[(m_2 + \beta K_2 - K_2)^2 - \alpha^2\right]}\left[K_3 - \frac{2K_4(m_2 + \beta K_2 - K_2)}{(m_2 + \beta K_2 - K_2)^2 - \alpha^2}\right]$$

$$K_{11} = -\frac{2Pr \exp(-K_1) K_4^2}{(1+N)\left[(2\beta - 1)^2 K_2^2 - \alpha^2\right]}$$

$$K_{12} = -\frac{2Pr \exp(-K_1) K_4}{(1+N)\left[(2\beta-1)^2 K_2^2 - \alpha^2\right]}\left[K_3 - \frac{2Pr K_2 K_4 (2\beta - 1)}{(2\beta-1)^2 K_2^2 - \alpha^2}\right]$$

$$K_{13} = -\frac{2Pr \exp(-K_1) C_1 K_4}{(1+N)\left[(m_1 + \beta K_2 - K_2)^2 - \alpha^2\right]},$$

$$K_{14} = -\frac{2Pr \exp(-K_1) C_2 K_4}{(1+N)\left[(m_1 + \beta K_2 - K_2)^2 - \alpha^2\right]}$$

$$K_{15} = -K_5 \exp[-(2m_1 - K_2)] - K_6 \exp[-(2m_2 - K_2)] - (K_7 + K_{11} - K_{12})$$
$$\exp[-(2\beta - 1)]K_2 - K_8 \exp[-(m_1 + m_2 - K_2)] - (K_9 - K_{13})$$
$$\exp[-(m_1 + \beta K_2 - K_2)] - (K_{10} - K_{14}) \exp[-(m_2 + \beta K_2 - K_2)]$$

$$K_{16} = -K_5 \exp(2m_1 - K_2) - K_6 \exp(2m_2 - K_2) - (K_7 + K_{11} + K_{12})$$
$$\exp(2\beta - 1)K_2 - K_8 \exp(m_1 + m_2 - K_2) - (K_9 - K_{13})$$
$$\exp(m_1 + \beta K_2 - K_2) - (K_{10} + K_{14}) \exp(m_2 + \beta K_2 - K_2),$$

$$C_3 = \frac{K_{16}\exp(\alpha) - K_{15}\exp(-\alpha)}{\exp(2\alpha) - \exp(-2\alpha)}$$

$$C_4 = \frac{K_{15}\exp(\alpha) - K_{16}\exp(-\alpha)}{\exp(2\alpha) - \exp(-2\alpha)}$$

$$m_1 = \frac{\beta K_2 + \sqrt{\beta^2 K_2^2 + 4M}}{2},$$

$$m_2 = \frac{\beta K_2 - \sqrt{\beta^2 K_2^2 - 4M}}{2}$$

$$K_{17} = \frac{\alpha \beta m_1 C_1 C_3}{(\alpha + m_1)(\alpha + m_1 - \beta K_2) - M^2},$$

$$K_{18}=\frac{\alpha\beta m_1C_1C_4}{(\alpha-m_1)(\alpha-m_1+\beta K_2)-M^2},$$

$$K_{19}=\frac{\alpha\beta m_2C_2C_3}{(\alpha+m_2)(\alpha+m_2-\beta K_2)-M^2},$$

$$K_{20}=\frac{\alpha\beta m_2C_2C_4}{(\alpha-m_2)(\alpha-m_2+\beta K_2)-M^2},$$

$$K_{21}=\frac{\alpha\beta C_3\left[(\beta K_2K_3+K_4)\left\{\alpha(\alpha+\beta K_2)-M^2\right\}-\beta K_2K_4(2\alpha+\beta K_2)\right]}{\left[\alpha(\alpha+\beta K_2)-M^2\right]^2}$$

$$K_{22}=\frac{\alpha\beta C_4\left[(\beta K_2K_3+K_4)\left\{\alpha(\alpha-\beta K_2)-M^2\right\}+\beta K_2K_4(2\alpha-\beta K_2)\right]}{\left[\alpha(\alpha-\beta K_2)-M^2\right]^2}$$

$$K_{23}=\frac{\beta m_1C_1K_5(2m_1-K_2)}{(3m_1-K_2)(3m_1-K_2-\beta K_2)-M^2},$$

$$K_{24}=\frac{\beta m_2C_2K_6(2m_2-K_2)}{(3m_2-K_2)(3m_2-K_2-\beta K_2)-M^2},$$

$$A_1=(\beta K_2K_3+K_4)[K_{12}+(2\beta-1)K_2K_7],$$

$$A_2=m_1C_1K_6(2m_2-K_2)+m_2C_2K_8(m_1+m_2-K_2),$$

$$A_3=m_2C_2K_5(2m_1-K_2)+m_1C_1K_8(m_1+m_2-K_2),$$

$$A_4=m_1C_1[K_{12}+K_2K_7(2\beta-1)]$$
$$+(\beta K_2K_3+K_4)[K_{13}+K_9(m_1+\beta K_2-K_2)],$$

$$A_5=m_2C_2[K_{12}+K_2K_7(2\beta-1)]$$
$$+(\beta K_2K_3+K_4)[K_{14}+K_{10}(m_2+\beta K_2-K_2)],$$

$$A_6=m_1C_1[K_{13}+K_9(m_1+\beta K_2-K_2)]$$
$$+K_5(\beta K_2K_3+K_4)(2m_1-K_2),$$

$$A_7=m_2C_2[K_{14}+K_{10}(m_2+\beta K_2-K_2)]$$
$$+K_6(\beta K_2K_3+K_4)(2m_2-K_2),$$

$$A_8=m_1C_1[K_{14}+K_{10}(m_2+\beta K_2-K_2)]$$
$$+m_2C_2[K_{13}+K_9(m_1+\beta K_2-K_2)]$$
$$+K_8(\beta K_2K_3+K_4)(m_1+m_2-K_2),$$

$$A_9=m_1C_1[2K_{11}+K_2K_{12}(2\beta-1)]$$
$$+K_{13}(\beta K_2K_3+K_4)(m_1+\beta K_2-K_2)$$
$$+\beta K_2K_4[K_{13}+K_9(m_1+\beta K_2-K_2)],$$

$$A_{10}=m_2C_2[2K_{11}+K_2K_{12}(2\beta-1)]$$
$$+K_{14}(\beta K_2K_3+K_4)(m_2+\beta K_2-K_2)$$
$$+\beta K_2K_4[K_{14}+K_{10}(m_2+\beta K_2-K_2)],$$

$$A_{11}=m_1C_1K_{13}(m_1+\beta K_2-K_2)+\beta K_2K_4K_5(2m_1-K_2)$$

$$A_{12}=m_2C_2K_{14}(m_2+\beta K_2-K_2)+\beta K_2K_4K_6(2m_2-K_2)$$

$$A_{13}=m_1C_1K_{14}(m_2+\beta K_2-K_2)$$
$$+m_2C_2K_{13}(m_1+\beta K_2-K_2)+\beta K_2K_4K_8(m_1+m_2-K_2)$$

$$A_{14}=(\beta K_2K_3+K_4)[2K_{11}+K_2K_{12}(2\beta-1)]$$
$$+\beta K_2K_4[K_{12}+K_2K_7(2\beta-1)],$$

$$A_{15}=m_1C_1K_2K_{11}(2\beta-1)+\beta K_2K_4K_{13}(m_1+\beta K_2-K_2)$$

$$A_{16}=m_2C_2K_2K_{11}(2\beta-1)+\beta K_2K_4K_{14}(m_2+\beta K_2-K_2)$$

$$A_{17}=K_2K_{11}(2\beta-1)(\beta K_2K_3+K_4)+\beta K_2K_4[2K_{11}$$
$$+K_2K_{12}(2\beta-1)],$$

$$A_{18}=\beta K_2^2K_4K_{11}(2\beta-1),$$

$$A_{19}=C_1m_1\exp(m_1)+C_2m_2\exp(m_2)$$
$$+(\beta K_2K_3+K_4)\exp(\beta K_2)+\beta K_2K_4\exp(\beta K_2),$$

$$A_{20}=C_5m_1\exp(m_1)+C_6m_2\exp(m_2)+K_{17}(\alpha+m_1)\exp(\alpha+m_1)$$
$$-K_{18}(\alpha-m_1)\exp[-(\alpha-m_1)]+K_{19}(\alpha+m_2)\exp(\alpha+m_2)$$
$$-K_{20}(\alpha-m_2)\exp[-(\alpha-m_2)]+[K_{21}(\alpha+\beta K_2)$$
$$+K_{39}(\alpha+\beta K_2)]\exp(\alpha+\beta K_2)+[K_{40}-(K_{22}+K_{40})$$
$$(\alpha-\beta K_2)]\exp(\alpha-\beta K_2)+[K_{53}-(K_{74}+K_{53})(\alpha+K_2)]\exp(\alpha-K_2)$$
$$[K_{54}-(K_{75}+K_{54})(\alpha-K_2)]\exp[-(\alpha-K_2)]+K_{23}(3m_1-K_2)$$
$$\exp(3m_1-K_2)+K_{24}(3m_2-K_2)\exp(3m_2-K_2)+[K_{38}+(3\beta-1)K_2$$
$$\{K_{25}+K_{38}+K_{43}+K_{44}\}+2K_{43}+3K_{44}]\exp(3\beta-1)K_2$$

$$+[K_{26}(m_1+2m_2-K_2)]\exp(m_1+2m_2-K_2)+K_{27}(2m_1+m_2-K_2)$$
$$\exp(2m_1+m_2-K_2)+[(K_{28}+K_{33}+K_{41})(m_1+2\beta K_2-K_2)+2K_{41}+K_{33}]$$
$$\exp(m_1+2\beta K_2-K_2)+[(K_{29}+K_{34}+K_{42})(m_2+2\beta K_2-K_2)+2K_{42}+K_{34}]$$
$$\exp(m_2+2\beta K_2-K_2)+[(K_{30}+K_{35})(2m_1+\beta K_2-K_2)+K_{35}]$$
$$\exp(2m_1+\beta K_2-K_2)+[(K_{31}+K_{36})(2m_2+\beta K_2-K_2)+K_{36}]$$
$$\exp(2m_2+\beta K_2-K_2)+[(K_{32}+K_{37})(m_1+m_2+\beta K_2-K_2)+K_{37}]$$
$$\exp(m_1+m_2+\beta K_2-K_2)+[2K_{76}m_1+K_{55}(1+2m_1)]\exp(2m_1)$$
$$+[2K_{77}m_2+K_{56}(1+2m_2)]\exp(2m_2)+[2\beta K_2K_{78}+K_{82}(1+2\beta K_2)$$
$$+K_{85}(2+2\beta K_2)+K_{64}(3+2\beta K_2)]\exp(2\beta K_2)$$
$$+[(m_1+m_2)K_{79}+K_{58}\{1+(m_1+m_2)\}]\exp(m_1+m_2)$$
$$+[K_{80}(m_1+\beta K_2)+K_{83}\{1+(m_1+\beta K_2)\}$$
$$+K_{62}\{2+(m_1+\beta K_2)\}]\exp(m_1+\beta K_2)+[K_{81}(m_2+\beta K_2)$$
$$+K_{84}\{1+(m_2+\beta K_2)\}+K_{63}\{2+(m_2+\beta K_2)\}]\exp(m_2+\beta K_2)$$

$$A_{21}=C_1m_1\exp(-m_1)+C_2m_2\exp(-m_2)$$
$$+(\beta K_2K_3+K_4)\exp(-\beta K_2)-\beta K_2K_4\exp(-\beta K_2),$$

$$A_{22}=C_5m_1\exp(-m_1)+C_6m_2\exp(-m_2)+K_{17}(\alpha+m_1)$$
$$\exp[-(\alpha+m_1)]-K_{18}(\alpha-m_1)\exp(\alpha-m_1)+K_{19}(\alpha+m_2)$$
$$\exp[-(\alpha+m_2)]-K_{20}(\alpha-m_2)\exp(\alpha-m_2)+[K_{21}(\alpha+\beta K_2)$$
$$+K_{39}+K_{39}(1+\alpha\beta K_2)]\exp[-(\alpha+\beta K_2)]+[-K_{22}$$
$$(\alpha-\beta K_2)+K_{40}+(\alpha-\beta K_2)K_{40}]\exp(\alpha-\beta K_2)$$
$$+[K_{74}(\alpha-K_2)+K_{53}-K_{53}(\alpha+K_2)]\exp[-(\alpha+K_2)]y$$
$$+[-K_{75}(\alpha-K_2)+K_{54}+K_{54}(\alpha-K_2)]\exp(\alpha+K_2)$$
$$+K_{23}(3m_1-K_2)\exp[-(3m_1-K_2)]+K_{24}(3m_2-K_2)$$
$$\exp[-(3m_2-K_2)y]+[(K_{25}-K_{38}+K_{43}-K_{44})$$
$$(3\beta-1)K_2-2K_{43}-3K_{44}+K_{38}]\exp[-(3\beta-1)K_2]$$
$$+K_{26}(m_1+2m_2-K_2)\exp[-(m_1+2m_2-K_2)]$$
$$+K_{27}(2m_1+m_2-K_2)\exp[-(2m_1+m_2-K_2)]$$
$$+[(K_{28}-K_{33}-K_{41})(m_1+2\beta K_2-K_2)-2K_{41}$$
$$+K_{33}]\exp[-(m_1+2\beta K_2-K_2)]+[(K_{29}-K_{34}+K_{42})$$
$$(m_2+\beta K_2-K_2)-2K_{42}+K_{34}]\exp[-(m_2+2\beta K_2-K_2)]$$
$$+[(K_{30}-K_{35})(2m_1+\beta K_2-K_2)+K_{35}]\exp[-(2m_1+\beta K_2-K_2)]$$
$$+[(K_{31}-K_{36})(2m_2+\beta K_2-K_2)+K_{36}]\exp[-(2m_2+\beta K_2-K_2)]$$
$$+[(K_{32}-K_{37})(m_1+m_2+\beta K_2-K_2)+K_{37}]\exp[-(m_1+m_2$$
$$+\beta K_2-K_2)]+[2m_1K_{76}+K_{55}(1-2m_1)]\exp(-2m_1)+[2m_2K_{77}$$
$$+K_{56}(1-2m_2)]\exp(-2m_2)+[2\beta K_2K_{78}+K_{82}(1-2\beta K_2)$$
$$+K_{85}(-2+2\beta K_2)+K_{64}(3-2\beta K_2)]\exp(-2m_2)+[(m_1+m_2)K_{79}$$
$$+K_{58}\{1-(m_1+m_2)\}]\exp[-(m_1+m_2)]+[K_{80}(m_1+\beta K_2)$$
$$+K_{83}\{1-(m_1+\beta K_2)\}+K_{62}\{-2+(m_1+\beta K_2)\}]$$
$$\exp[-(m_1+\beta K_2)]+[K_{81}(m_2+\beta K_2)+K_{84}\{1-(m_2+\beta K_2)\}$$
$$+K_{63}\{-2+(m_2+\beta K_2)\}]\exp[-(m_2+\beta K_2)],$$

$$K_{25}=\frac{A_1\beta}{\left[K_2^2(3\beta-1)(2\beta-1)-M^2\right]}-\frac{\beta[A_4K_2(5\beta-1)+2A_{17}]}{\left[K_2^2(3\beta-1)(2\beta-1)-M^2\right]^2}$$
$$+\frac{2\beta K_2(5\beta-1)[K_2A_{17}(5\beta-1)+6A_{18}]}{\left[K_2^2(3\beta-1)(2\beta-1)-M^2\right]^3}-\frac{6A_{18}\beta K_2^3(5\beta-1)^3}{\left[K_2^2(3\beta-1)(2\beta-1)-M^2\right]^4}$$

$$K_{26}=\frac{A_2\beta}{(m_1+2m_2-K_2)(m_1+2m_2-\beta K_2-K_2)-M^2},$$

$$K_{27}=\frac{A_3\beta}{(2m_1+m_2-K_2)(2m_1+m_2-\beta K_2-K_2)-M^2},$$

$$K_{28}=\frac{A_4\beta}{(m_1+2\beta K_2-K_2)(m_1+\beta K_2-K_2)-M^2}$$
$$-\frac{\beta[A_9(2m_1+3\beta K_2-2K_2)+2A_{15}]}{\left[(m_1+2\beta K_2-K_2)(m_1+\beta K_2-K_2)-M^2\right]^2}$$
$$+\frac{2A_{15}\beta(2m_1+3\beta K_2-2K_2)^2}{\left[(m_1+2\beta K_2-2K_2)(m_1+\beta K_2-K_2)-M^2\right]^3},$$

$$K_{29}=\frac{A_5\beta}{(m_2+2\beta K_2-K_2)(m_2+\beta K_2-K_2)-M^2}$$
$$-\frac{\beta[A_{10}(2m_2+3\beta K_2-2K_2)+2A_{16}]}{\left[(m_2+2\beta K_2-K_2)(m_2+\beta K_2-K_2)-M^2\right]^2}$$
$$+\frac{2A_{16}\beta(2m_2+3\beta K_2-K_2)^2}{\left[(m_2+\beta K_2-K_2)(m_2+\beta K_2-K_2)-M^2\right]^3},$$

$$K_{30}=\frac{A_6\beta}{(2m_1+\beta K_2-K_2)(2m_1-K_2)-M^2}-\frac{A_{11}\beta(4m_1+\beta K_2-K_2)}{\left[(2m_1+\beta K_2-K_2)(2m_1-K_2)-M^2\right]^2},$$

$$K_{31}=\frac{A_7\beta}{(2m_2+\beta K_2-K_2)(2m_2-K_2)-M^2}-\frac{A_{12}\beta(4m_2+\beta K_2-K_2)}{\left[(2m_2+\beta K_2-K_2)(2m_2-K_2)-M^2\right]^2},$$

$$K_{32}=\frac{A_8\beta}{(m_1+m_2+\beta K_2-K_2)(m_1+m_2-K_2)-M^2}-\frac{A_{13}\beta(2m_1+2m_2+\beta K_2-2K_2)}{\left[(m_1+m_2+\beta K_2-K_2)(m_1+m_2-K_2)-M^2\right]^2},$$

$$K_{33}=\frac{A_9\beta}{\left[(2m_1+2\beta K_2-K_2)(m_1+\beta K_2-K_2)-M^2\right]}-\frac{2A_{15}\beta(2m_1+3\beta K_2-2K_2)}{\left[(m_1+2\beta K_2-K_2)(m_1+\beta K_2-K_2)-M^2\right]^2},$$

$$K_{34}=\frac{A_{10}\beta}{\left[(m_2+2\beta K_2-K_2)(m_2+\beta K_2-K_2)-M^2\right]}-\frac{2A_{16}\beta(2m_2+3\beta K_2-2K_2)}{\left[(m_2+2\beta K_2-K_2)(m_2+\beta K_2-K_2)-M^2\right]^2},$$

$$K_{35}=\frac{A_{11}\beta}{\left[(2m_1+\beta K_2-K_2)(2m_1-K_2)-M^2\right]},$$

$$K_{36}=\frac{A_{12}\beta}{\left[(2m_2+\beta K_2-K_2)(2m_2-K_2)-M^2\right]},$$

$$K_{37}=\frac{A_{13}\beta}{\left[(m_1+m_2+\beta K_2-K_2)(m_1+m_2-K_2)-M^2\right]},$$

$$K_{38}=\frac{A_{14}\beta}{K_2^2(3\beta-1)(2\beta-1)-M^2}-\frac{\beta[A_{14}+6A_{18}+2A_{17}K_2(5\beta-2)]}{\left[K_2^2(3\beta-1)(2\beta-1)-M^2\right]^2}+\frac{6A_{18}\beta K_2^2(5\beta-1)^2}{\left[K_2^2(3\beta-1)(2\beta-1)-M^2\right]^3}$$

$$K_{39}=\frac{\alpha\beta^2K_2K_4C_3}{\alpha(\alpha+\beta K_2)-M^2},$$

$$K_{40}=\frac{\alpha\beta^2K_2K_4C_4}{\alpha(\alpha-\beta K_2)-M^2},$$

$$K_{41}=\frac{A_{15}\beta}{\left[(m_1+2\beta K_2-K_2)(m_1+\beta K_2-K_2)-M^2\right]},$$

$$K_{42}=\frac{A_{16}\beta}{\left[(m_2+2\beta K_2-K_2)(m_2+\beta K_2-K_2)-M^2\right]},$$

$$K_{43}=\frac{A_{17}\beta}{K_2^2(3\beta-1)(2\beta-1)-M^2}-\frac{3A_{18}\beta K_2(5\beta-2)}{\left[K_2^2(3\beta-1)(2\beta-1)-M^2\right]^2}$$

$$K_{44}=\frac{A_{18}\beta}{K_2^2(3\beta-1)(2\beta-1)-M^2},$$

$$K_{45}=\frac{\beta\exp(K_1)C_3}{\left[(\alpha+K_2)(\alpha-\beta K_2+K_2)-M^2\right]}\left\{(P-GrK_1)+\frac{GrK_2(2\alpha+2K_2-\beta K_2)}{\left[(\alpha+K_2)(\alpha-\beta K_2+K_2)-M^2\right]}\right\},$$

$$K_{46}=\frac{\beta\exp(K_1)C_4}{\left[(\alpha-K_2)(\alpha+\beta K_2-K_2)-M^2\right]}\left\{(P-GrK_1)-\frac{GrK_2(2\alpha-2K_2+\beta K_2)}{\left[(\alpha-K_2)(\alpha+\beta K_2-K_2)-M^2\right]}\right\},$$

$$K_{47}=\frac{\beta\exp(K_1)K_5}{\left(4m_1^2-2\beta K_2m_1-M^2\right)}[(P-GrK_1)$$

$$+\frac{GrK_2(4m_1-\beta K_2)}{\left(4m_1^2-2\beta K_2 m_1-M^2\right)}\Bigg],$$

$$K_{48}=\frac{\beta\exp(K_1)K_6}{\left(4m_2^2-2\beta K_2 m_2-M^2\right)}\Bigg[(P-GrK_1)$$

$$+\frac{GrK_2(4m_2-\beta K_2)}{\left(4m_2^2-2\beta K_2 m_2-M^2\right)}\Bigg],$$

$$K_{49}=\frac{18\beta^2\exp(K_1)GrK_2^2K_{11}\left(5\beta^2K_2^2+2M^2\right)}{\left(2\beta^2K_2^2-M^2\right)^4}$$

$$+\frac{2\beta\exp(K_1)\left(7\beta^2K_2^2+M^2\right)[K_{11}(P-GrK_1)-GrK_1K_{12}]}{\left(2\beta^2K_2^2-M^2\right)^3}$$

$$-\frac{3\beta^2\exp(K_1)K_2[K_{12}(P-GrK_1)-GrK_2K_7]}{\left(2\beta^2K_2^2-M^2\right)^2}$$

$$-\frac{\beta\exp(K_1)K_7(P-GrK_1)}{\left(2\beta^2K_2^2-M^2\right)},$$

$$K_{50}=\frac{\beta\exp(K_1)K_8}{\left[(m_1+m_2)(m_1+m_2-\beta K_2)-M^2\right]}$$

$$\left\{1+\frac{GrK_2(2m_1+2m_2-\beta K_2)}{(m_1+m_2)(m_1+m_2-\beta K_2)-M^2}\right\},$$

$$K_{51}=\frac{\beta\exp(K_1)K_9(P-GrK_1)}{\left[m_1(m_1+\beta K_2)-M^2\right]}$$

$$-\frac{\beta\exp(K_1)(2m_1+\beta K_2)[K_{13}(P-GrK_1)-GrK_2K_9]}{\left[m_1(m_1+\beta K_2)-M^2\right]^2}$$

$$-\frac{2\beta\exp(K_1)GrK_2K_{13}\left(3m_1^2+\beta^2K_2^2+3m_1\beta K_2+M^2\right)}{\left[m_1(m_1+\beta K_2)-M^2\right]^3}$$

$$K_{52}=\frac{\beta\exp(K_1)K_{10}(P-GrK_1)}{\left[m_2(m_2+\beta K_2)-M^2\right]}$$

$$-\frac{\beta\exp(K_1)(2m_2+\beta K_2)[K_{14}(P-GrK_1)-GrK_2K_{10}]}{\left[m_2(m_2+\beta K_2)-M^2\right]^2}$$

$$-\frac{2\beta\exp(K_1)GrK_2K_{14}\left(3m_2^2+\beta^2K_2^2+3m_2\beta K_2+M^2\right)}{\left[m_2(m_2+\beta K_2)-M^2\right]^3}$$

$$K_{53}=-\frac{Gr\beta\exp(K_1)K_2C_3}{\left[(\alpha+K_2)(\alpha-\beta K_2+K_2)-M^2\right]},$$

$$K_{54}=-\frac{Gr\beta\exp(K_1)K_2C_4}{\left[(\alpha-K_2)(\alpha+\beta K_2-K_2)-M^2\right]},$$

$$K_{55}=-\frac{\beta\exp(K_1)GrK_2K_5}{\left(4m_1^2-2m_1\beta K_2-M^2\right)},$$

$$K_{56}=-\frac{\beta\exp(K_1)GrK_2K_6}{\left(4m_2^2-2m_2\beta K_2-M^2\right)},$$

$$K_{57}=\frac{\beta\exp(K_1)[K_{12}(P-GrK_1)-GrK_2K_7]}{\left[2\beta^2K_2^2-M^2\right]}$$

$$-\frac{6\beta^2K_2\exp(K_1)[K_{11}(P-GrK_1)-GrK_2K_{12}]}{\left[2\beta^2K_2^2-M^2\right]^2}$$

$$-\frac{6\beta\exp(K_1)GrK_2K_{11}\left(7\beta^2K_2^2+M^2\right)}{\left[2\beta^2K_2^2-M^2\right]^3},$$

$$K_{58}=-\frac{\beta\exp(K_1)GrK_2K_8}{\left[(m_1+m_2)(m_1+m_2-\beta K_2)-M^2\right]},$$

$$K_{59}=\frac{\beta\exp(K_1)[K_{13}(P-GrK_1)-GrK_2K_9]}{\left[m_1(m_1+\beta K_2)-M^2\right]}+\frac{2\beta\exp(K_1)GrK_2K_{13}(2m_1+\beta K_2)}{\left[m_1(m_1+\beta K_2)-M^2\right]^2},$$

$$K_{60}=\frac{\beta\exp(K_1)[K_{14}(P-GrK_1)-GrK_2K_{10}]}{\left[m_2(m_2+\beta K_2)-M^2\right]}+\frac{2\beta\exp(K_1)GrK_2K_{14}(2m_2+\beta K_2)}{\left[m_2(m_2+\beta K_2)-M^2\right]^2},$$

$$K_{61}=\frac{\beta\exp(K_1)\left(2\beta^2K_2^2-M^2\right)[K_{11}(P-GrK_1)-GrK_2K_{12}]}{\left(2\beta^2K_2^2-M^2\right)^2}+\frac{9\beta^2K_2^2\exp(K_1)GrK_{11}}{\left(2\beta^2K_2^2-M^2\right)^2},$$

$$K_{62}=-\frac{\beta\exp(K_1)GrK_2K_{13}}{\left[m_1(m_1+\beta K_2)-M^2\right]},$$

$$K_{63}=-\frac{\beta\exp(K_1)GrK_2K_{14}}{\left[m_2(m_2+\beta K_2)-M^2\right]},$$

$$K_{64}=-\frac{\beta\exp(K_1)GrK_2K_{11}}{\left[2\beta^2K_2^2-M^2\right]},$$

$$K_{65}=-\frac{Gr\beta\exp(K_1)C_3}{\left[(\alpha+K_2)(\alpha-\beta K_2-K_2)-M^2\right]},$$

$$K_{66}=-\frac{Gr\beta\exp(K_1)C_4}{\left[(\alpha-K_2)(\alpha+\beta K_2-K_2)-M^2\right]},$$

$$K_{67}=-\frac{Gr\beta\exp(K_1)K_5}{\left(4m_1^2-2\beta K_2m_1-M^2\right)},$$

$$K_{68}=-\frac{Gr\beta\exp(K_1)K_6}{\left(4m_2^2-2\beta K_2m_2-M^2\right)},$$

$$K_{69}=\frac{Gr\beta\exp(K_1)}{\left(2\beta^2K_2^2-M^2\right)}\left[-K_7+\frac{3\beta K_2K_{12}}{\left(2\beta^2K_2^2-M^2\right)}-\frac{2K_{11}\left(7\beta^2K_2^2+M^2\right)}{\left(2\beta^2K_2^2-M^2\right)^2}\right],$$

$$K_{70}=\frac{Gr\beta\exp(K_1)}{\left[m_1(m_1+\beta K_2)-M^2\right]}\left\{-K_9+\frac{K_{13}(2m_1+\beta K_2)}{\left[m_1(m_1+\beta K_2)-M^2\right]}\right\}$$

$$K_{71}=\frac{Gr\beta\exp(K_1)}{\left[m_2(m_2+\beta K_2)-M^2\right]}\left\{-K_{10}+\frac{K_{14}(2m_2+\beta K_2)}{\left[m_2(m_2+\beta K_2)-M^2\right]}\right\}$$

$$K_{72}=\frac{Gr\beta\exp(K_1)}{\left(2\beta^2K_2^2-M^2\right)}\left[-K_{12}+\frac{6\beta K_2K_{11}}{\left(2\beta^2K_2^2-M^2\right)}\right],$$

$$K_{73}=-\frac{Gr\exp(K_1)\beta K_8}{\left[(m_1+m_2)(m_1+m_2-\beta K_2)-M^2\right]}$$

$$K_{74}=K_{45}+K_{65},\qquad K_{75}=K_{46}+K_{66},$$

$$K_{76}=K_{47}+K_{67},\qquad K_{77}=K_{48}+K_{68},$$

$$K_{78}=K_{49}+K_{69},\qquad K_{79}=K_{50}+K_{73},$$

$$K_{80}=K_{51}+K_{70},\qquad K_{81}=K_{52}+K_{71},$$

$$K_{82}=K_{57}+K_{72},\qquad K_{83}=K_{59}+\frac{K_{62}}{K_2},$$

$$K_{84} = K_{60} + \frac{K_{63}}{K_2}, \qquad K_{85} = K_{61} + \frac{K_{64}}{K_2},$$

$$\begin{aligned} K_{86} = & -K_{17}\exp(\alpha+m_1) - K_{18}\exp[-(\alpha-m_1)] - K_{19}\exp(\alpha+m_2) \\ & -K_{20}\exp[-(\alpha-m_2)] - K_{21}\exp(\alpha+\beta K_2) - K_{22}\exp[-(\alpha-\beta K_2)] \\ & -K_{23}\exp(3m_1-K_2) - K_{24}\exp(3m_2-K_2) - K_{25}\exp(3\beta-1) \\ & -K_{26}\exp(m_1+2m_2-K_2) - K_{27}\exp(2m_1+m_2-K_2) \\ & -K_{28}\exp(m_1+2\beta K_2-K_2) - K_{29}\exp(m_2+\beta K_2-K_2) \\ & -K_{30}\exp(2m_1+\beta K_2-K_2) - K_{31}\exp(2m_2+\beta K_2-K_2) \\ & -K_{32}\exp(m_1+m_2+\beta K_2-K_2) - K_{33}\exp(m_1+2\beta K_2-K_2) \\ & -K_{34}\exp(m_2+2\beta K_2-K_2) - K_{35}\exp(2m_1+\beta K_2-K_2) \\ & -K_{36}\exp(2m_2+\beta K_2-K_2) - K_{37}\exp(m_1+m_2+\beta K_2-K_2) \\ & -K_{38}\exp(3\beta-1)K_2 - K_{39}\exp(\alpha+\beta K_2) - K_{40}\exp[-(\alpha-\beta K_2)] \\ & -K_{41}\exp(m_1+2\beta K_2-K_2) - K_{42}\exp(m_2+2\beta K_2-K_2) \\ & -K_{43}\exp(3\beta-1)K_2 - K_{44}\exp(3\beta-1)K_2 - K_{53}\exp(\alpha+K_2) \\ & -K_{54}\exp[-(\alpha-K_2)] - K_{55}\exp(2m_1) - K_{56}\exp(2m_2) \\ & -K_{58}\exp(m_1+m_2) - K_{62}\exp(m_1+\beta K_2) \\ & -K_{63}\exp(m_2+\beta K_2) - K_{64}\exp(2\beta K_2) \\ & -K_{74}\exp(\alpha+K_2) - K_{75}\exp[-(\alpha-K_2)] \\ & -K_{76}\exp(2m_1) - K_{77}\exp(2m_2) - K_{78}\exp(2\beta K_2) \\ & -K_{79}\exp(m_1+m_2) - K_{80}\exp(m_1+\beta K_2) - K_{81} \\ & \exp(m_2+\beta K_2) - K_{82}\exp(2\beta K_2) - K_{83}\exp(m_1+\beta K_2) \\ & -K_{84}\exp(m_2+\beta K_2) - K_{85}\exp(2\beta K_2), \end{aligned}$$

$$\begin{aligned} K_{87} = & -K_{17}\exp[-(\alpha+m_1)] - K_{18}\exp(\alpha-m_1) - K_{19} \\ & \exp[-(\alpha+m_2)] - K_{20}\exp(\alpha-m_2) - K_{21}\exp[-(\alpha+\beta K_2)] \\ & -K_{22}\exp(\alpha-\beta K_2) - K_{23}\exp[-(3m_1-K_2)] - K_{24} \\ & \exp[-(3m_2-K_2)] - K_{25}\exp[-(3\beta-1)K_2] - K_{26} \\ & \exp[-(m_1+2m_2-K_2)] - K_{27}\exp[-(2m_1+m_2-K_2)] \\ & -K_{28}\exp[-(m_1+2\beta K_2-K_2)] - K_{29}\exp[-(m_2+2\beta K_2-K_2)] \\ & -K_{30}\exp[-(2m_1+\beta K_2-K_2)] - K_{31}\exp[-(2m_2 \\ & +\beta K_2-K_2)] - K_{32}\exp[-(m_1+m_2+\beta K_2-K_2)] \\ & -K_{33}\exp[-(m_1+2\beta K_2-K_2)] - K_{34}\exp[-(m_2 \\ & +2\beta K_2-K_2)] - K_{35}\exp[-(2m_1+\beta K_2-K_2)] \\ & -K_{36}\exp[-(2m_2+\beta K_2-K_2)] - K_{37}\exp[-(m_1 \\ & +m_2+\beta K_2-K_2)] - K_{38}\exp[-(3\beta-1)]K_2 \\ & -K_{39}\exp[-(\alpha+\beta K_2)] - K_{40}\exp(\alpha-\beta K_2) \\ & -K_{41}\exp[-(m_1+2\beta K_2-K_2)] - K_{42}\exp[-(m_2+2\beta K_2 \\ & -K_2)] - K_{43}\exp[-(3\beta-1)]K_2 - K_{44}\exp[-(3\beta-1)]K_2 \\ & -K_{53}\exp[-(\alpha+K_2)] - K_{54}\exp(\alpha-K_2) - K_{55}\exp(-2m_1) \\ & -K_{56}\exp(-2m_2) - K_{58}\exp[-(m_1+m_2)] - K_{62}\exp[-(m_1 \\ & +\beta K_2)] - K_{63}\exp[-(m_2+\beta K_2)] - K_{64}\exp[-(2\beta K_2)] \\ & -K_{74}\exp[-(\alpha+K_2)] - K_{75}\exp(\alpha-K_2) - K_{76}\exp(-2m_1) \\ & -K_{77}\exp(-2m_2) - K_{78}\exp(-2\beta K_2) - K_{79}\exp[-(m_1 \\ & +m_2)] - K_{80}\exp[-(m_1+\beta K_2)] - K_{81}\exp[-(m_2+\beta K_2)] \\ & -K_{82}\exp[-(2\beta K_2)] - K_{83}\exp[-(m_1+\beta K_2)] \\ & -K_{84}\exp[-(m_2+\beta K_2)] - K_{85}\exp(-2\beta K_2) \end{aligned}$$