Ultra Scientist Vol. 24(1)B, 75-80 (2012).

Unified analytic study of stellar structure

SUDHANSHU PANDEY[1] and K. L. PANDEY[2]

[1]Department of Physics, Nehru Gram Bharti University, Kotwa, Dubawal, Allahabad (INDIA)
[2]Department of Physics, Ewing Christian College, Allahabad (INDIA)

(Acceptance Date 31st January, 2012)

Abstract

Polytropes are self gravitating gaseous sphere that are very useful as crude approximation to more realistic stellar modal. In this paper we have demonstrated that the density/mass of the stars keeps on increasing from its surface to centre. Approximate analytic solutions to the equilibrium equations have been presented in phase planes such as (ξ_θ, θ), Transformations connecting solutions in this phase plane have been obtained and discussed.

Key word: Polytropes, stellar modal.

Introduction

The behavior of solutions of the Lane-Emden equations is polytropic index n, which controls the distribution of physical variables, has been studied by Hopf [1], Fowler[2] and Chandrasekhar for n<3, n=3, and n>3, respectively. It is well known so far from some of these studies that the polytropic index n=0 and 1 represent, the liquid and gaseous states of a polytrope of uniform density respectively. The origin and the behavior of Lane-Emden equations were reported same whatever be the index of a polytrope[3-14]. The Miline[14] was able to determine the maximum limiting density[15] and the maximum value of mass of a star[16] for $n \to 0$ and $n \to 1$ whereas the structure of planet was also reported[17,18] for the same values of n. Further thermo dynamical equilibrium of stars clusters embedded in an isothermal configuration[19], relativistic stellar structures and X-ray transients in Ni's theory of gravity[20], very massive stellar models in Ni's theory of gravity[21], and general relativity neutron star[22] were also reported for the same values of n.

The theory of polytropes in which the pressure (P) and density (ρ) are related by a monomomial relation of the kind, $P = K\rho^{1+\frac{1}{n}}$ (n and K are two disposable constants; n is the polytropic index, and K defines the temperature implicitly) may be considered as a fundamental parameters to the study of stellar structures.

Considering the stars, which are in

equilibrium and in a steady state can be characterized by three physical parameters *i.e.* its mass *M*; its radius *R*; and its luminosity *L*(*L* refers to the amount of radiant energy in ergs, radiated by the star per second to the space outside) analytic series solutions to the equilibrium equations have been presented in phase planes such as (ξ_θ, θ), Transformations connecting solutions in this phase plane have been obtained, Since the nucleus includes the immediate neighborhood of the origin (n=0), it will be of the interest to investigate it, in the light of this new concept of uniform density for $n \to 0$ and $n \to 1$.

Structure Equation in (ξ_θ, θ) phase plane :

The equations governing the structure of a polytopic configuration of index n with angular velocity Ω can be expressed with the help of electromagnetic Maxwall's equations

$$\frac{P}{\rho} = \nabla\varphi + \frac{1}{2}\Omega^2 X^2,\ X^2 = x^2 + y^2 \qquad (1)$$

$$P = K\rho^{1+\frac{1}{n}} \qquad (2)$$

$$\nabla^2 \varphi = -4\pi G\rho \qquad (3)$$

where, P is the pressure, ρ the density, ϕ the gravitational potential, *X* the distance from the axis of rotation, K a constant, and G the gravitational constant (6.67 x 10^{-8} dynes cm^2/ gm^2). If we introduce Υ as the distance from the centre of the polytrope, and define the dimensionless variable θ, and ξ_θ by the relations

$$\rho = \rho_c\theta^n ; \gamma = \alpha_\theta\xi_\theta = \left[\frac{(n+1)k}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{1/2}\xi_\theta \qquad (4)$$

$$\omega = \frac{\Omega^2}{2\pi G\rho_c}$$

where ρ_c is the central density.

From equations (1), (2) & (3) we can deduce the following expression:

$$\frac{1}{\xi_\theta^N}\frac{d}{d\xi_\theta}\left(\xi_\theta^N\frac{d\theta}{d\xi_\theta}\right) = -\theta + \omega \qquad (5)$$

Which satisfied the boundary conditions

$$\theta = 1,\ \frac{d\theta}{d\xi_\theta} = 0 \quad \text{at} \quad \xi_\theta = 0 \qquad (6)$$

Equation (5) is known as general "Lane-Emden" Equation for polytropic index n. For the convenience we take,

$$\xi_\theta = \xi \qquad (7)$$

Also equation (5) can be written as,

$$\frac{N}{\xi}\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2} = -\theta^n + \omega \qquad (8)$$

For non-rotating case, $\omega = 0$ as $\Omega = 0$

$\therefore$Equation (8) becomes

$$\frac{N}{\xi}\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2} = -\theta^n \qquad (9)$$

Equation (9) is the required structure equation, for non-rotating case, in (ξ_θ, θ) phase plane.

Here we solve the structure Equation for polytropic index n = 1 and N=2 (spheroidal), (N = 1) (cylindrical) and for N = 0, (plane symmetric)

Case-1 for n=1 and N=2, equation (9) becomes,

$$\frac{2}{\xi}\frac{d\theta}{d\xi}+\frac{d^2\theta}{d\xi^2}=-\theta^n \qquad (10)$$

applying the boundary conditions

$$\text{for } \theta = 1, \ \frac{d\theta}{d\xi} = 0 \text{ at } \xi = 0 \qquad (11)$$

The series solution of the form, satisfying the boundary conditions can be expressed as $\theta=1 + a_1\xi^2 + a_2\xi^4 + a_3\xi^6 + a_4\xi^8 + \ldots\ldots\ldots\ldots$

or $\theta = 1 + a_1\xi^2 + a_2\xi^4 + a_3\xi^6 + a_4\xi^8 + a_5\xi^{10} + a_6\xi^{12} + \ldots\ldots\ldots$ (12)

Differentiating equation (12) w. r. t. ξ we get

$$\frac{d\theta}{d\xi}=2a_2\xi+4a_2\xi^3+6a_3\xi^5+8a_4\xi^7+10a_5\xi^9+\ldots\ldots\ldots \qquad (13)$$

again differentiating equation (13) w.r.t. ξ we get,

$$\frac{d^2\theta}{d\xi^2}=2a_1+12a_2\xi^2+30a_3\xi^4+56a_4\xi^6+90a_5\xi^8+132a_6\xi^{10}+\ldots\ldots\ldots \qquad (14)$$

From equations (10), (12), (13) & (14) we get

$$\frac{2}{\xi}\left(2a_1\xi+4a_2\xi^3+6a_3\xi^5+8a_4\xi^7+10a_5\xi^9+12a_6\xi^{11}+\ldots..\right)$$
$$+\left(2a_1+129a_2\xi^2+30a_3\xi^4+56a_4\xi^6+90a_5\xi^8+132a_6\xi^{10}+\ldots..\right)$$
$$=-(1+a_1\xi^2+a_2\xi^4+a_3\xi^6+a_4\xi^8+a_5\xi^{10}+a_6\xi^{12}+\ldots\ldots)$$

Equating the co-efficient of powers of ξ, we get,

$6a_1 = -1 \Rightarrow a_1 = -\frac{1}{6}$

$20a_2 = a1 \Rightarrow a_2 = \frac{1}{120}$

$43a_3 = -a_2 \Rightarrow a_3 = -\frac{1}{5040}$

$74a_4 = -a_3 \Rightarrow a_4 = -\frac{1}{362880}$

$110a_5 = -a_4 \Rightarrow a_5 = -\frac{1}{39916800}$

$156a_6 = -a_5 \Rightarrow a_6 = -\frac{1}{6227020800}$

Putting the Value of a_1, a_2, a_3, a_4, a_5 and a_6 in equation (12) we get, the required solution.

$$\theta=1-\frac{1}{6}\xi^2+\frac{1}{120}\xi^4-\frac{1}{5040}\xi^6+\frac{1}{362880}\xi^8\ldots\ldots\ldots\ldots \qquad (15)$$

Case-2 for cylindrical shape i.e. N = 1 and n = 1, equation (9) becomes

$$\frac{1}{\xi}\frac{d\theta}{d\xi}+\frac{d^2\theta}{d\xi^2}=-\theta \qquad (16)$$

series solution can be expressed as

$$\theta(\xi)=1+a_1\xi^2+a_2\xi^4+a_3\xi^6+a_4\xi^8+a_5\xi^{10}+\ldots\ldots\ldots \qquad (17)$$

Differentiating above w. r. t. ξ

$$\frac{d\theta}{d\xi}=2a_1\xi+4a_2\xi^3+6a_3\xi^5+8a_4\xi^7+10a_5\xi^9+\ldots\ldots\ldots \qquad (18)$$

again differentiating above w. r. t. ξ

$$\frac{d\theta^2}{d\xi^2}=2a_1+12a_1\xi^2+30a_3\xi^4+56a_4\xi^6+90a_5\xi^8+\ldots\ldots\ldots \qquad (19)$$

putting these values in equation (16)

$$\frac{1}{\xi}\left(2a_1\xi+4a_2\xi^3+6a_3\xi^5+8a_4\xi^7+10a_5\xi^9+..........\right)$$
$$+\left(2a_1+12a_2\xi^2+30a_3\xi^4+56a_4\xi^6+90a_5\xi^5+..........\right)$$
$$=-\left(1+a_1\xi^2+a_2\xi^4+a_3\xi^6+a_4\xi^8+a_5\xi^{10}+..........\right)$$

equating the co-efficient of powers of ξ.

$4a_1 = -1 \quad \Rightarrow \quad a_1 = -\frac{1}{4}$

$16a_2 = -a_1 \quad \Rightarrow \quad a_2 = -\frac{1}{64}$

$36a_3 = -a_2 \quad \Rightarrow \quad a_3 = -\frac{1}{2304}$

$100a_5 = -a_4 \quad \Rightarrow \quad a_5 = -\frac{1}{147,45600}$

substituting the value of constants a_1, a_2, a_3, a_4 and a_5 in equation (17) we get

$$\theta=1-\frac{1}{4}\xi^2+\frac{1}{64}\xi^4-\frac{1}{2304}\xi^6+\frac{1}{147,456}\xi^8 -\frac{1}{147,45600}\xi^{10}+......... \tag{20}$$

Case-3 for N =0 & n=1, equation (9) becomes,

$$\frac{d^2\theta}{d\xi^2}=-\theta \tag{21}$$

equation (21) is in the form of well known, simple harmonic (SHM) motion equation
Solution of above equation (21) is given by

$$\theta = a_1\sin\xi + a_2\cos\xi$$

$$\frac{d\theta}{d\xi} = a_1\cos\xi - a_2\sin\xi$$

putting the boundary conditions in above equation *i.e.* θ = 1, $\frac{d\theta}{d\xi}=0$ at ξ = 0, we get

$a_1 = 0$ and $a_2 = 1$

$$\theta = \cos\xi \tag{22}$$

Results and Discussion

Graphical representation of (ξ, θ) plane for N =2 & n=1 (Fig. a), for N =1 & n=1 (Fig. b) and N=0 & n=1 (Fig. c), where ξ show radius and θ show density of polytropes. The graphs plotted by our series solution method are in good agreement by the graph with the stellar model. It is evident from the figure that the as radius of the polytropes decreases, its density increases in all the three cases implying that the mass of the stars keeps on increasing as we move from surface to centre. The graph for N=0, N=1, and N=2 between θ & ξ has been plotted and found to be in good agreement with the results graph of N=0 (plane Symmetric) N=1 (Cylindrical) N=2(spheroidal) the shape stellar structure of given value.

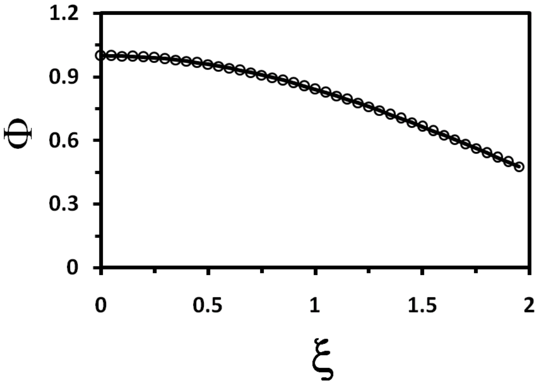

Fig a. Graphical representation of (ξ, θ) phase plane for N =2 & n=1 where ξ show radius and θ show density of polytropes, from the figure that the radius of the polytropes decreases, its density increases, mass is more and more centrally condensed, the density of the polytropes never fall below zero.

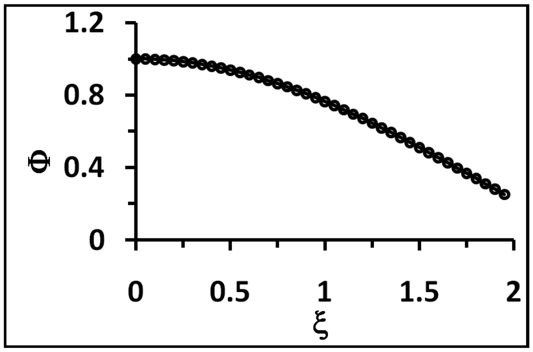

Fig. b. Graphical representation of (ξ, θ) phase plane for N =1 & n=1 where ξ show radius and θ show density of polytropes, from the figure that the radius of the polytropes decreases, its density increases, mass is more and more centrally condensed, the density of the polytropes never fall below zero.

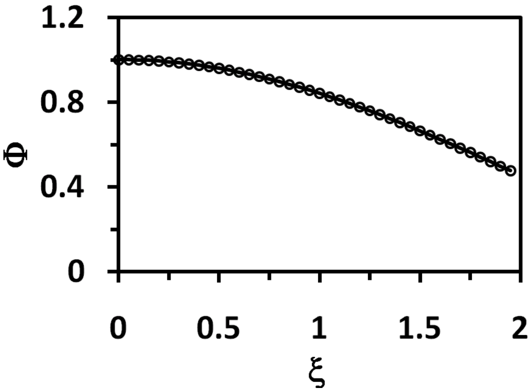

Fig. c. Graphical representation of (ξ, θ) phase plane for N =0 & n=1 where ξ show radius and θ show density of polytropes, from the figure that the radius of the polytropes decreases, its density increases, mass is more and more centrally condensed, the density of the polytropes never fall below zero.

Conclusions

An unified analytic study structure of the nucleons of Polytropes N=0 (Plane Symmetric) N=1 (Cylindrical) N=2 (spheroidal) has been investigated following the concept of sphere of uniform density defined by polytropic index (n) tending to zero. The graphs plotted by our series solution method are in good agreement by the graph with the stellar model. The mass of the stars keeps on increasing as we move from surface to centre. Our given analysis can be applied to the interdisciplinary modeling, environmental and biological systems which may quite often involve complicated forms of linear or non-linear differential equation.

References

1. E. Hopf, On solutions of Lane-Emden Equation for $1 < n \leq 3$, MNRAS, 91 635-662 (1932).
2. R.H. Fowler and Grid properties of E-solutions an $\xi \rightarrow 0$, MNRAS, 91 63-71 (1930).
3. J. Homer Lane, *American Journal of Science,* 2dser, 50, 57, 1869
4. S. Chandrasekhar, An Introduction to the Study Stellar structure, Dover Publications, Inc., N.Y., (1939).
5. D. Menzel, P.L. Bhatnagar and H.K. Sen, Stellar Interiors, Chapman and Hall, London, Chap. 10, (1963).
6. J. Ostriker, Ap. J., 140, 1056 , 1964; also.*J. Suppl.,* 11, 167, 1965.
7. Shambhunath Srivastava and J.P. Sharma: *Experientia, 24,* 764 (1968).
8. E.R. Harrison and R.G. Lake, *Ap. J.,* 171, 323, (1972).
9. J.P. Sharma, Thesis of D. Phil., Univ. of Allahabad (U.P.), (1970).
10. J.P. Sharma, Thesis of D.Sc., Univ. of Gorakhpur (U.P.), India (1981).

11. J.P. Sharma, Proc; Math . Soc., *BHU,* 16, (2000).
12. Indira Bardoloi *et al., Bull. Astr., India, 32,* 1, (2004).
13. J.P. Sharma, Relativistic Stellar Structures and X-Ray Transients in Ni's Theory of Gravity, 76th Ann. Conf., Natn. Acad. Sc., India (Presented), 2006; and Proc. Natn. Acad., Sc., India communicated for publication (2006).
14. E. A. Miline, Structure of self-gravitating polytropic sphere, *Hdb D ap, 3,* 65-71 (1930).
15. J.P. Sharma, Limiting density in while dwarfs, *Indian J. Phys, 48,* 66-70 (1974).
16. J.P. Sharma, On the maximum value of the mass of star, Trans New York Acad Sc, 34, 691-694 (1972).
17. J.P. Sharma, Planetary structures in general relativity, PAGEOPH, *97*, 14-24 (1972).
18. R.A. Lyttleton, On the origin of mountains, Proc Roy Soc, *275,* 1-11 (1963).
19. J.P. Sharma, Thermodynamically equilibrium of star cluster, Proc Nat Acad Sc India, *76,* 305-307 (2006).
20. J.P. Sharma, Relativistic stellar structures and X-ray Transients in Ni' theory of gravity. Proc Nat Acad Sc. Inida, *79*, 29-34 (2009).
21. J.P. Sharma, D.P. Jayapandian and Sanish Thomas, Very massive stellar models in Ni's theory of gravity, Proc Nat Acad Sc India, *79*, 411-416 (2009).
22. Sanish Thomas and J.P. Sharma, General relativistic neutron stars: Pramana (communicated) (2009).