

Strongly W-projective Module & Strongly W-injective

BIRESH KUMAR GUPTA, GOVIND SAHU and M.R. ALONEY

Babulal Tarabai Inst. of Research and Tech. Sagar and TIT, Bhopal (INDIA)

(Acceptance Date 15th February, 2012)

Abstract

In this paper the notions of strongly W-projective and strongly W-injective module dimension over commutative domain R are given. It is shown that over semi simple rings with W-dimension one, these modules are equivalent to W-injective modules. The W-projective dimension measures how far away a dimension is from being a Prüfer domain.

Introduction

In this note, R will denote a commutative domain with identity and $Q(\neq R)$ will denote its field of quotients. The R-module Q/R will be denoted by K. Lee² studied the structure of weak injective modules.

A R-module M is called strongly injective if $Ext_R^1(N, M^J) = 0$ for all R-modules N of weak dimension ≤ 1 . In section 2 we introduce a class of R-modules under the name of strongly W-projective R-modules, we show that Strongly W-projective modules R-modules are identical to projective R-modules iff R is semi simple. Recall that R is called Prüfer domain if every finitely generated ideal of R is projective. We show that each strongly W-projective R-module is FP - projective.

(I) *Strongly W-projective* :

Definition 1.1: A R-module M is called strongly W-projective if $Ext_R^1(N, M^J) = 0$ for every weak injective R-module N and J is index set, where $M^J = \bigoplus_{i \in J} M_i$

Evidently, direct products and summands of strongly W-projective R-modules are again strongly W-projective. All projective R-modules are trivially strongly W-projective but the converse is not true. For example Q/Z is a Z-module is strongly W-projective but it is not projective. Over a semi simple ring R. W-projective R-modules are projective. It is obvious that if R is a semi simple ring with $W.D(R) = 1$, then every R-module M is strongly W-projective iff M is strongly W-injective. Also, if R is semi simple and M is a strongly W-projective R-module, then $Ext_R^1(M^J, R) = 0$.

A well-known result states that an R-module F is Flat iff its character module F^J

is injective.

Lemma (1.2) : If R is a Noetherian ring and M is strongly W -projective R -module then M is FP-projective

Proof: Let M be a strongly W -projective, we must prove that $Ext_R^1(M^J, N) = 0$, for any FP-injective R -module N .

Since R is a Noetherian ring, N is an injective R -module, therefore N is strongly W -injective.

Lemma (1.3): Let $O \rightarrow A \rightarrow B \rightarrow C \rightarrow O$ is an exact sequence such that A and C are strongly W -projective R -module then B is strongly W -projective R -module then B is strongly W -projective.

Proof: Let N is a strongly W -injective R -module, from the induced exact sequence $Ext_R^1(C, N) \rightarrow Ext_R^1(B, N) \rightarrow Ext_R^1(A, N)$. We have $Ext_R^1(B, N) = 0$. Since $Ext_R^1(C, N) = Ext_R^1(A, N) = 0$ where NCM^J this implies $Ext_R^1(C, M^J) \rightarrow Ext_R^1(B, M^J) \rightarrow Ext_R^1(A, M^J)$ we have $Ext_R^1(B, M^J) = 0$ since $Ext_R^1(C, M^J) = Ext_R^1(A, M^J) = 0$

Example (1.4) : Z -module Q is strongly W -projective Recall that R is called of matlis domain if the projective dimension of Q (or equivalently K) is one. The R -module C is called matlis cotorsion if $Ext_R^1(Q, C) = 0$ and M is called strongly flat if $Ext_R^1(M^J, C) = 0$ for every matlis cotorsion R -module C .

Lemma (1.5) : If R is a matlis domain and M is a strongly flat R -module, then M is strongly W -projective.

Proof : If M is strongly flat R -module, then $Ext_R^1(M^J, N) = 0$ for all matlis cotorsion R -modules N . It is easy to see that if R is a matlis domain then every strongly W -projective R -module is matlis cotorsion.

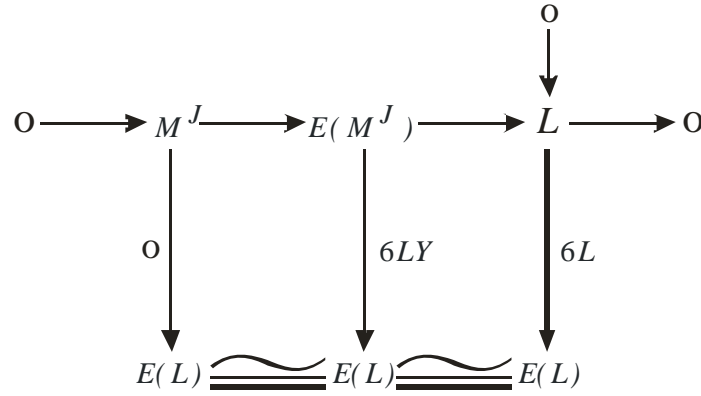
Lemma (4) : Let R is a semi Dedekind domain. If M is a projective R -module and N is a strongly W -projective R -module, then $M^J \otimes_R N$ is strongly W -projective.

Proof: The isomorphism $Tor_n^R(M^J \otimes N, A) \cong (M^J, A) = 0$ for all torsion free R -module A . The converse is true when R is a local semi Dedekind domain.

In what follows $\sigma M : M^J \rightarrow E(M^J)$ denotes the injective envelope of an R -module M . Recall that an injective envelope $\sigma M : M^J \rightarrow E(M^J)$ has the unique mapping property if for my homomorphism $f : M^J \rightarrow N$ with N injective there exists a unique homomorphism $g : E(M^J) \rightarrow N$ such that $g\sigma M = f$.

Proposition (1.6): Every Strongly W -injective R -module has an injective envelope with the unique mapping then R is a priifer domain.

Proof: Let M be any strongly W -injective R -module we have the following exact commutative diagram.



Note: That $\sigma L \gamma \sigma M = 06m$, So $\sigma L \gamma = 0$ by given condition, therefore $L = \text{im}(\gamma) \subset \text{ker } \sigma L = 0$ and hence M is injective.

The following characterization of strongly W-projective R-modules. Let λ be a class of R-modules and M is a R-module A homomorphism & $\phi \in \text{HOM}_R(N, M^J)$ with $N \phi l$ is called an l -procover of M^J if the induced map.

$\text{HOM}_R(1_N, \phi) : \text{HOM}_R(N^1, N) \rightarrow \text{HOM}_R(N^1, M^J)$ is subjective for all $N^1 \in l$. An l -procover $\phi \in \text{HOM}_R(N, M^J)$ is called an l -cover if each $\gamma \in \text{HOM}_R(N, N)$ satisfying $\phi = \phi \gamma$ is an automorphism of N. the class l is called a precover (cover) class if every R-module has an l -precover (l -cover)

The l -pre-envelope, l - envelope, pre-envelope and envelope classes are defined dually. In particular, if l is the class of strongly W-injective R-modules an l - envelope is called a strongly W-injective envelope.

Proposition (1.7) : If M is an R-

module, then the following are equivalent:

- (a) M is strongly W-projective
- (b) M is projective with respect to every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where A is strongly W-injective.
- (c) For every exact sequence $0 \rightarrow K \rightarrow F \rightarrow M^J \rightarrow 0$ where F is strongly W-injective $K \rightarrow F$ is a strongly W-injective pre-envelope of K.
- (d) M is co-kernel of a strongly W-injective pre-envelope $K \rightarrow F$ with F projective.

Proof: (a) \Rightarrow (b) Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence where A is strongly W-injective then $\text{Ext}_R^1(M^J, A) = 0$ by (a).

Thus $\text{HOM}_R(M^J, B) \rightarrow \text{HOM}_R(M^J, C) \rightarrow 0$ is exact and (b) holds.

(b) \Rightarrow (a) For every strongly W-injective R-module N, there is a short exact sequence $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$ with E injective, which implies an exact sequence $\text{HOM}_R(M^J, E) \rightarrow \text{HOM}_R(M^J, L) \rightarrow \text{Ext}_R^1(M^J, N) \rightarrow 0$. Since $\text{HOM}_R(M^J,$

- $E) \rightarrow \text{HOM}_R(M^J, L) \rightarrow 0$ is exact by
 (b) we have $\text{Ext}_R^1(M^J, N) = 0$ and (a) follows (a) \Rightarrow (c) it is easy to verify
 (c) \Rightarrow (d) Let $0 \rightarrow K \rightarrow P \rightarrow M^J \rightarrow 0$ be an exact sequence with P-projective. Note that P is strongly W-injective by hypothesis thus $K \rightarrow P$ is a strongly W-injective pre-envelope.
 (d) \Rightarrow (a) By (d) there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M^J \rightarrow 0$ where $K \rightarrow P$ is strongly W-injective pre-envelope with P projective. It gives to the exactness of $\text{HOM}_R(P, N) \rightarrow \text{HOM}_R(K, N) \rightarrow \text{Ext}_R^1(M^J, N) \rightarrow 0$ for each strongly W-injective R-module N. Note that $\text{HOM}_R(P, N) \rightarrow \text{HOM}_R(K, N) \rightarrow 0$ is exact by (d). Hence $\text{Ext}_R^1(M^J, N) = 0$ as desired.

(II) *Strongly W-injective dimension :*

Definition (2.0) (a) For any R-module M, let strongly W-injective $\text{wid}(M^J)$ of M, denote the smallest integer $n \geq 0$ such that $\text{Ext}_R^1(n, M^J) = 0$ for every R-module N of W-dimension ≤ 1 . (If no such n exists set $\text{wid}(m) = \infty$)
 (b) $\text{wid}(R) = \text{Sup} \{ \text{wid}(M^J) : M \text{ is an R-module} \}$
Lemma (2.1): Let R is a semi-Dedekind domain for R-module M. If $\text{wid}(M^J) \leq n$ then $\text{Ext}_R^1(N, M^J) = 0$ for all R-module N of W-dimension ≤ 1

Proof : Use induction on n clearly if $\text{wid}(M^J) = n$ If $\text{wid}(M^J) \leq n - 1$ resolve

N by $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$ with K and P that K hence W-dimension ≤ 1 by and $\text{Ext}_R^{n+1}(N, M^J) \cong \text{Ext}_R^1(K, M^J) = 0$ by induction hypothesis.

Definition (2.2): For an R-module M, let $\text{swpd}(M^J)$ denotes the smallest integer $n \geq 0$ such that $\text{Ext}_R^{n+1}(M^J, N) = 0$ for every W-injective R-module N and call $\text{swpd}(M^J)$ the W-projective dimension of M. If no such n exists $\text{swpd}(M^J) = \infty$ put $\text{rswpd}(H) = \text{Sup} \{ \text{rswpd}(M^J) : M \text{ is an right R-module} \}$ and call $\text{rswpd}(R)$ the right strongly W-projective dimension of R, M is called strongly W-projective if $\text{swpd}(M^J) = 0$ i.e. $\text{Ext}_R(M^J, N) = 0$ for every W-injective R-module N.

Remark (2.3) : For every right R and every R-module M, the inequalities $\text{SWPD}(R) \leq \text{D}(R)$ and $\text{swpd}(M^J) \leq \text{Spd}(M^J)$ are valid. It is easy to see that $\text{swpd}(M^J) = \text{spd}(M^J)$ for any R-module M iff every W-projective R-module is projective.

Lemma (2.4) : Let R is a semi Dedekind domain for R-module M, if $\text{Ext}_R^{n+1}(M^J, N) = 0$ for any W-injective R-module N then $\text{Ext}_R^{n+j}(M^J, N) = 0$ for any W-injective R-module N for $j \geq 1$.

Proof: For any W-injective R-module N, there is a short exact sequence $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$ where E is injective. Then the sequence $\text{Ext}_R^{n+1}(M^J, L) \rightarrow \text{Ext}_R^{n+2}(M^J, N) \rightarrow \text{Ext}_R^{n+2}(M^J, E) = 0$ is exact. Note that L

is strongly W-injective by Lemma 2.1 So $Ext_R^{n+1}(M^J, L) = 0$ by given condition. Hence $Ext_R^{n+2}(M^J, N) = 0$.

Proposition (2.5) : Let R be a semi Dedekind domain, if $SWPD(R) \leq 1$ then the every sub module of a strongly W-projective R-module is strongly W-projective.

Proof: Let N be a sub-module of a strongly W-projective R-module M then for any strongly W-injective R-module L we get an exact sequence.

$$0 = Ext_R^1(M^J, N) \rightarrow Ext_R^1(N, L) \rightarrow Ext_R^2\left(\frac{M^J}{N}, L\right)$$

Note: that the last term is zero by given condition. Hence $Ext_R^1(N, R) = 0$

Theorem (2.6): Let R is a semi Dedekind domain then $\sup\{Spd(M^J) : M \text{ is W-projective R-module}\} \leq Swid(R)$.

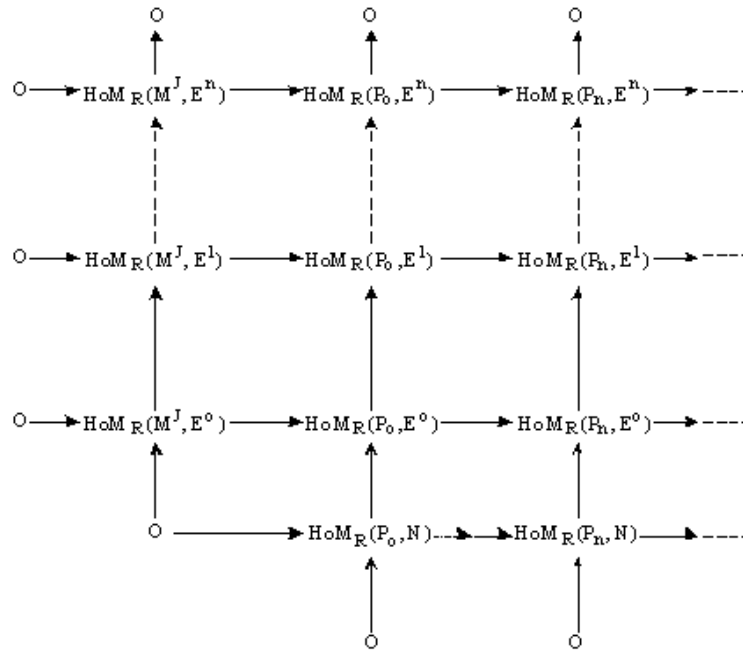
Proof: Let M is a strongly W-projective R-module. It is enough to show that $Spd(M^J) \leq swid(R)$. We may assume that $swid(R) \leq \infty$. M admits a projective resolution

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M^J \rightarrow 0$$

Let N be any R-module we have $swid(N) \leq n$ Thus by lemma 2.1 there is an exact sequence.

$$0 \cdots \rightarrow N \rightarrow E^0 \rightarrow E^1 \cdots \rightarrow E^{n-1} \rightarrow E^n \rightarrow 0$$

Where $E^0, E^1, E^2, \dots, E^n$ are strongly W-injective. Therefore we form a double complex.



Note that all rows are exact except for the bottom row, since M is strongly W -projective and all E^i are strongly W -injective also note that all columns are exact except for all left column

since all P_i are projective.

Using a spectral sequence argument, we know that the too complexes

$$0 \rightarrow \text{HOM}_R(P_0, N) \rightarrow \text{HOM}_R(P_1, N) \rightarrow \dots \rightarrow \text{HOM}_R(P_n, N) \rightarrow \text{and}$$

$$0 \rightarrow \text{HOM}_R(M^J, E^0) \rightarrow \text{HOM}_R(M^J, E^1) \rightarrow \dots \rightarrow \text{HOM}_R(M^J, E^n) \rightarrow 0$$

have isomorphic homology groups. Thus $\text{Ext}_R^{n+j}(M^J, N) = 0$ for all $j \geq 1$ Hence $\text{Spd}(M^J) \leq n$ It is known that $D(R) = \sup\{\text{Spd}(M^J) : M \text{ is a } W\text{-projective } R\text{-module}\}$ if R is a priifer domain and it is easy to see that $D(R) = \text{SWPD}(R)$ if R is a semi simple ring.

References

1. Ding, N., On envelope with the unique Map. *Prop comm. Alg.* 24(4) 1459-1470 (1996).
2. Lee. S.B. Weak-injective modules. *Comm. Alg.* 361-370 (2006).
3. Mao, L. and Ding N. Q., Notes on FP-projective modules and FP-injective modules, *Advance in ring theory* 151- 166 (2005).
4. Rotman, J.J., *An introduction to Homological Algebra* New York; Academic Press (1979).
5. Smith. P. F., Injective Modules and Prime ideals *Comm. Alg.* 9, 989-999 (1981).
6. Trlifaj, J., covers, Envelope and cotorsion Theories, *Lecture Notes for the workshop Homological Methods in modules Theory Cortona* (2000).