

## Polytropes and their property

SUDHANSHU PANDEY<sup>1</sup> and K. L. PANDEY<sup>2</sup>

(Acceptance Date 21st November, 2012)

### Abstract

Polytropes has been analytical studied under the concept of a sphere of uniform density by the polytropic indices  $n$  tending to zero and  $n$  tending one. In this paper we have demonstrated that the temperature of the stars keeps on decreasing from its centre to surface. Approximate analytic solutions to the equilibrium equations have been presented in phase planes such as  $(U\rho, V\rho)$ , Transformations connecting solutions in this phase plane have been obtained and discussed.

*Key word* : Polytropes, Temperature , Analytical Study, Phase plane.

### Introduction

The structure of a self-gravitating equilibrium polytropic sphere is described by Lane-Emden equation which have been used to model several phenomena in mathematical physics, thermodynamics, astrophysics such as the theory of stellar structure and thermal behavior of a spherical cloud of gas, for the value of polytropic index  $n$  between 0 and 3, the equation approximates to a reasonable accuracy the structure of a variety of the realistic stellar models. Though modern texts no longer give them through treatment that classical works of Emden and Chandrasekhar do, Closed form analytical solution, has been studied by Fowler<sup>1</sup>, Hopf<sup>2</sup> and Chandrasekhar for  $n < 3$ ,  $n = 3$ , and  $n > 3$ , respectively. It is well known so

far from some of these studies that the polytropic index  $n=0$  and 1 represent, the liquid and gaseous states of a polytrope of uniform density respectively. The origin and the behavior of Lane-Emden equations were reported same whatever be the index of a polytrope<sup>3-14</sup>. The Miline<sup>3</sup> was able to determine the maximum limiting density<sup>15</sup>, whereas the structure of planet was also reported<sup>16,17</sup> for the same values of  $n$ , and the maximum value of mass of a star<sup>18</sup> for  $n \rightarrow 0$  and  $n \rightarrow 1$ , general relativity neutron star<sup>19</sup> were also reported for the same values of  $n$ , and very massive stellar models in Ni's theory of gravity<sup>20</sup>, relativistic stellar structures and X-ray transients in Ni's theory of gravity<sup>21</sup>, Further thermo dynamical equilibrium of stars clusters embedded in an isothermal

configuration<sup>22</sup>.

Considering the stars, which are in equilibrium and in a steady state can be characterized by three physical parameters *i.e.* its mass  $M$ ; its radius  $R$ ; and its luminosity  $L$  ( $L$  refers to the amount of radiant energy in ergs, radiated by the star per second to the space outside,) analytic series solutions to the equilibrium equations have been presented in phase planes such as  $(U\rho, V\rho)$ , Transformations connecting solutions in this phase plane have been obtained, Since the nucleus includes the immediate neighborhood of the origin ( $n=0$ ), it will be of the interest to investigate it, in the light of this new concept of temperature of polytropes.

*Structure Equation in  $(V\rho, V\rho)$  phase plane:*

Lane-Emden equation is written in the form of the generalized equation in  $(\xi\rho, \rho)$  phase plane is given by

$$\frac{1}{\xi\rho^N} \frac{d}{d\xi\rho} \left( \xi\rho^N \rho^{\frac{1}{n}-1} \frac{d\rho}{d\xi\rho} \right) = -\rho + \rho_c \omega \quad (1)$$

$$\text{where } r = \alpha_\rho \xi\rho \quad \& \quad \alpha_\rho = \left[ \frac{K(n+1)}{4\pi mG} \right]^{\frac{1}{2}}$$

for non-rotating polytropes  $\omega=0$  and polytropic index  $n=1$ ,

Equation (1) becomes :

$$\frac{1}{\xi\rho^N} \frac{d}{d\xi\rho} \left( \xi\rho^N \frac{d\rho}{d\xi\rho} \right) = -\rho$$

$$\text{or, } \frac{N}{\xi\rho} \frac{d\rho}{d\xi\rho} + \frac{d^2\rho}{d\xi\rho^2} = -\rho \quad (2)$$

We define  $U\rho$  and  $V\rho$  for non-rotating case and taking polytropic index  $n=1$ , as

$$U\rho = \frac{-\xi\rho\rho}{\rho'} \quad (3a)$$

$$V\rho = \frac{-\xi\rho\rho'}{\rho} \quad (3b)$$

$$\text{Where } \rho' = \frac{d\rho}{d\xi\rho}$$

The First order equation between  $U\rho$ , &  $V\rho$  can be obtained as follows,

From equation 3 (a)

$$\frac{1}{U\rho} \frac{dU\rho}{d\xi\rho} = \frac{1}{\xi\rho} + \frac{\rho'}{\rho} - \frac{\rho''}{\rho'} \quad (4)$$

Also from equation (2), we can write.

$$\rho'' = -\rho - \frac{N}{\xi\rho} \rho' \quad (5)$$

Using (4) & (5) We have,

$$\begin{aligned} \frac{1}{U\rho} \frac{dU\rho}{d\xi\rho} &= \frac{1}{\xi\rho} \left[ 1 + \frac{\xi\rho\rho'}{\rho} - \frac{\xi\rho\rho''}{\rho'} \right] \\ &= \frac{1}{\xi\rho} \left[ 1 + \frac{\xi\rho\rho'}{\rho} - \frac{\xi\rho}{\rho'} \left\{ -\rho - \frac{N}{\xi\rho} \rho' \right\} \right] \end{aligned}$$

$$= \frac{1}{\xi_\rho} \left[ (1+N) + \frac{\xi_\rho \rho'}{\rho} + \frac{\xi_\rho \rho}{\rho'} \right] \quad \frac{dU_\rho}{dV_\rho} = - \frac{U_\rho}{V_\rho} \left[ \frac{U_\rho + V_\rho - (1+N)}{U_\rho + V_\rho + (1-N)} \right] \quad (9)$$

Using equation 3(a)

$$\frac{1}{U_\rho} \frac{dU_\rho}{d\xi_\rho} = \frac{1}{\xi_\rho} [(1+N) - V_\rho - U_\rho] \quad (6)$$

Now again we have from 3(b).

$$\frac{1}{V_\rho} \frac{dV_\rho}{d\xi_\rho} = \frac{1}{\xi_\rho} - \frac{\rho'}{\rho} + \frac{\rho''}{\rho'} \quad (7)$$

Using Equation (6) & (7) and by similar process as above we get

$$\frac{1}{V_\rho} \frac{dV_\rho}{d\xi_\rho} = \frac{1}{\xi_\rho} [(1-N) + U_\rho + V_\rho] \quad (8)$$

From equations (6) and (8) we get,

$$\frac{V_\rho}{U_\rho} \frac{dU_\rho}{dV_\rho} = - \left[ \frac{U_\rho + V_\rho - (1+N)}{U_\rho + V_\rho + (1-N)} \right]$$

or

$$\begin{aligned} &\Rightarrow \left( a_1 V_\rho + 2a_2 V_\rho^2 + 3a_3 V_\rho^2 + 4a_4 V_\rho^3 + \dots \right) \times (1+N + \\ &\quad \alpha_1 V_\rho + \alpha_2 V_\rho^3 + \alpha_3 V_\rho^4 + \alpha_4 V_\rho^4 + \dots + V_\rho + (1-N)) \\ &= - \left\{ (1+N) + a_1 V_\rho + a_2 V_\rho^2 + a_3 V_\rho^3 + a_4 V_\rho^4 + \dots \right\} \\ &\quad \left\{ (1+N) + \left( a_1 \frac{1}{2} \right) V_\rho + a_2 V_\rho^2 + a_3 V_\rho^3 + a_4 V_\rho^4 + \dots - (1+N) \right\} \\ &\Rightarrow 2\alpha_1 V_\rho + [a_1(a_1+1) + 4a_2] V_\rho^2 + [a_1 a_2 + 2a_2(a_1+1) + 6\alpha_3] \\ &\quad = - \left\{ (1+N)(a_1+1) \right\} + V_\rho \left\{ a_1(a_1+1) + (1+N)a_2 \right\} V_\rho^2 \end{aligned}$$

Which is the required structure equation in (V<sub>ρ</sub>, U<sub>ρ</sub>) phase plane.

We solve the structure equation by assuming series solution for three cases, as follows,

$$U_\rho = (1+N) + a_1 V_\rho + a_2 V_\rho^2 + a_3 V_\rho^3 + a_4 V_\rho^4 + \dots \quad (10)$$

We find that it satisfies the initial condition

$$U_\rho \rightarrow (1+N) \text{ \& } V_\rho \rightarrow 0 \text{ as } \xi_\rho \rightarrow 0$$

Differentiating equation (10) we get ;

$$\frac{dU_\rho}{dV_\rho} = a_1 + 2a_2 V_\rho + 3a_3 V_\rho^2 + 4a_4 V_\rho^3 + \dots \quad (11)$$

putting the value of U<sub>ρ</sub> from (10) &  $\frac{dU_\rho}{dV_\rho}$  from (11) in equation (9). we get,

$$\begin{aligned}
 & + - \left[ \left\{ (1+N)(3 + a_1 a_2 + a_2(a_1 + 1)) \right\} V_\rho^2 \right. \\
 & + \left\{ (1+N)(3 + \alpha_1 \alpha_2 + \alpha_2(\alpha_1 + 1)) \right\} V_\rho^3 \\
 & \left. + \left\{ (1+N)\alpha_4 + \alpha_1 \alpha_3 + \alpha_2^2 + \alpha_3(\alpha_1 + 1) \right\} V_\rho^4 \dots \dots \dots \right] \tag{12}
 \end{aligned}$$

Now we discuss about equation (12) in different cases :

Case I :- For Spheroidal shape : (N=2).

from equation (12),

$$\begin{aligned}
 & 2\alpha_1 V_\rho + \left\{ \alpha_1(\alpha_1 + 1) + 4\alpha_2 \right\} V_\rho^2 + \left[ \alpha_2 \alpha_2 + 2\alpha_2(\alpha_2 + 1) + 6\alpha_3 \right] V_\rho^3 \\
 & + \left[ a_1 a_3 + 8a_4 + 3a_3(a_1 + 1) + 2a_2^2 \right] V_\rho^4 + \dots \dots \dots \\
 & = - \left[ 3(a_1 + 1) + V_\rho + \left\{ a_1(a_1 + 1) + 3a_1 \right\} V_\rho^2 + \left\{ 3a_3 + a_1 a_2 + a_2(a_1 + 1) \right. \right. \\
 & \left. \left. V_\rho^2 + \left\{ 3a_4 + a_1 a_3 + a_2^2 + a_3(a_1 + 1) \right\} V_\rho^2 + \dots \dots \dots \right]
 \end{aligned}$$

Equating the co-efficient of  $V_\rho$ , we get

$$2a_1 = -3(a_1 + 1)$$

$$a_1 = -3/5$$

$$a_1(a_1 + 1) + 4a_2 = - \left\{ a_1(a_1 + 1) + 3a_2 \right\}$$

$$a_2 = \frac{12}{175}$$

$$a_1 a_2 + 2a_2(a_1 + 1) + 6a_3 = -3a_3 - a_1 a_2 - a_2(a_1 + 1)$$

$$a_3 = 0$$

$$a_1 a_3 + 8a_4 + 3a_3(a_1 + 1) + 2a_2^2 = -3a_4$$

$$-a_1 a_3 - a_2^2 - a_3(a_1 + 1)$$

$$a_4 = -\frac{432}{336875}$$

for N = 2, series solution becomes, from equation (10)

$$U_\rho = 3 - \frac{3}{5} V_\rho + \frac{12}{175} V_\rho^2 - \frac{432}{336875} V_\rho^4 + \dots \tag{13}$$

Case II For Cylindrical shape i.e. N=1.

From equation (12), we get

$$\begin{aligned}
 & 2a_1V_\rho + \{a_1(a_1 + 1) + 4a_2\}V_\rho^2 \\
 & + \{a_1a_2 + 2a_2(a_1 + 1) + 6a_3\} \\
 & V_\rho^3 + \{a_1a_3 + 8a_4 + 3a_3(a_1 + 1) + 2a_2^2\}V_\rho^4 + \dots \\
 & = -[2(a_1 + 1)V_\rho + \{a_1(a_1 + 1) + 2a_2\}V_\rho^2 \\
 & + \{2a_2 + a_1a_2 + a_2^2 \\
 & + a_3(a_1 + 1)\}V_\rho^3 + \{2a_4 + a_1a_3 + a_2^2 \\
 & + a_3(a_1 + 1)\}V_\rho^4 \dots\dots\dots
 \end{aligned}$$

Equating the co-efficient of powers of  $V_\rho$

$$\begin{aligned}
 2a_1 &= -2(a_1 + 1) \Rightarrow a_1 = -\frac{1}{2} \\
 a_1(a_1 + 1)4a_2 &= -a_1(a_1 + 1) - 2a_2 \\
 a_2 &= \frac{1}{12} \\
 a_1a_2 + 2a_2(a_1 + 1) + 6a_3 &= -2a_3 - a_1a_2 - a_2(a_1 - 1) \\
 8a_3 &= 2a_2a_2 - 3a_2(a_1 + 1) \\
 a_3 &= -\frac{1}{192} \\
 a_1a_3 + 8a_4 + 3a_3(a_1 + 1) + 2a_2^2 &= -2a_4 \\
 -a_1a_3 - a_2^2 - a_3(a_1 + 1) &
 \end{aligned}$$

$$a_4 = -\frac{3}{1920}$$

For  $N=1$ , series solution becomes, from equation (10).

$$U_\rho = 2 - \frac{1}{2}V_\rho + \frac{1}{12}V_\rho^2 - \frac{1}{192}V_\rho^3 - \frac{3}{1920}V_\rho^4 + \dots \quad (14)$$

Case III : For plane-symmetric shape *i.e.*  $N=0$  from equation (12), we get,

$$\begin{aligned}
 & 2a_1V_\rho + [a_1(a_1 + 1) + 4a_2]V_\rho^2 + [a_1a_2 \\
 & + 2a_2(a_1 + 1) + 6a_3]V_\rho^3 \\
 & + [a_1a_3 + 8a_4 + 3a_3(a_1 + 1) + 2a_2^2]V_\rho^4 + \dots \\
 & = -(a_1 + 1)V_\rho - \{a_1(a_1 + 1) + a_2\}V_\rho^2 - \{a_3 + a_1a_2 \\
 & + a_2(a_1 + 1)\}V_\rho^3 - \{a_4 + a_1a_3 + a_2^2 + a_3(a_1 + 1)\}V_\rho^4
 \end{aligned}$$

Equating the co-efficient of powers of  $V_\rho$

$$\begin{aligned}
 2a_1 &= -(a_1 + 1) \Rightarrow a_1 = -\frac{1}{3} \\
 a_1(a_1 + 1) + 4a_2 &= -a_1(a_1 + 1) - a_2 \\
 a_2 &= \frac{4}{45} \\
 a_1a_2 + 2a_2(a_1 + 1) + 6a_3 &= -a_3 - a_1a_2 - a_2(a_1 + 1) \\
 a_3 &= -\frac{64}{945} \\
 a_1a_3 + 8a_4 + 3a_3(a_1 + 1) + 2a_2^2 & \\
 = -a_4 - a_1a_3 - a_2^2 - a_3(a_1 + 1) & \\
 a_4 &= \frac{1584}{127575}
 \end{aligned}$$

For  $N=0$ , series solution becomes, from equation (10).

$$U_{\rho} = 1 - \frac{1}{3}V_{\rho} + \frac{4}{45}V_{\rho}^2 - \frac{64}{945}V_{\rho}^3 + \frac{1584}{127575}V_{\rho}^4 - \dots \quad (15)$$

**Results and Discussion**

Graphical representation of  $(U_{\rho}, V_{\rho})$ , phase plane for  $N=2$  &  $n=1$  (Fig. 1), for  $N=1$  &  $n=1$  (Fig. 2) and  $N=0$  &  $n=1$  (Fig. 3), where  $U_{\rho}$  show temperature and  $V_{\rho}$  show radius of polytropes. The graphs plotted by our series solution method are in good agreement by the graph with the stellar model<sup>3</sup>. It is evident from the figure that the as temperature of the polytropes increases, its radius decreases in all the three cases implying that the temperature of the stars keeps on increasing as we move from surface to centre. The graph for  $N=0$ ,  $N=1$ , and  $N=2$  between  $U_{\rho}$  &  $V_{\rho}$  has been plotted and found to be in good agreement with the results graph of  $N=0$  (plane Symmetric)  $N=1$  (Cylindrical)  $N=2$  (spheroidal) the shape stellar structure of given value.

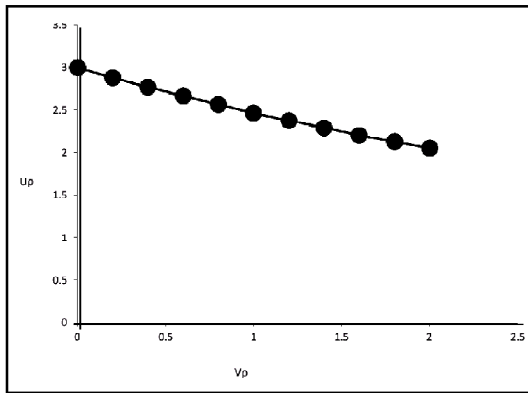


Fig. 1. Graphical representation of  $(U_{\rho}, V_{\rho})$  phase plane for  $N = 2$  &  $n = 1$  where  $V_{\rho}$  show radius,  $U_{\rho}$  show Temperature

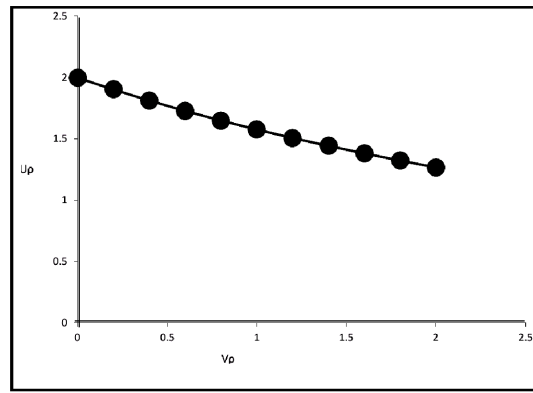


Fig. 2. Graphical representation of  $(U_{\rho}, V_{\rho})$  phase plane for  $N = 1$  &  $n=1$  where  $V_{\rho}$  show radius,  $U_{\rho}$  show Temperature

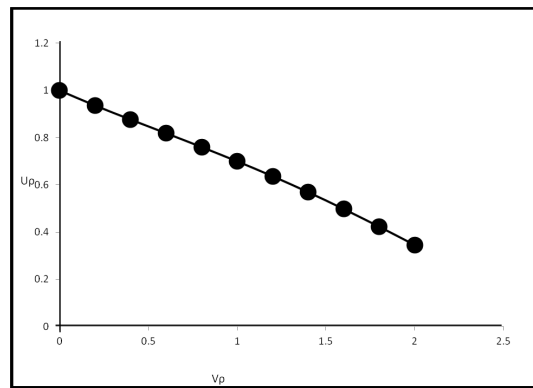


Fig. 3. Graphical representation of  $(U_{\rho}, V_{\rho})$  phase plane for  $N = 0$  &  $n=1$  where  $V_{\rho}$  show radius,  $U_{\rho}$  show Temperature

**Conclusions**

An unified analytic study structure of the nucleons of Polytropes  $N=0$  (Plane Symmetric)  $N=1$  (Cylindrical)  $N=2$  (spheroidal) has been investigated following the concept of sphere of uniform density defined by polytropic index ( $n$ ) tending to zero. The graphs

plotted by our series solution method are in good agreement by the graph with the stellar model<sup>3</sup>. The temperature of the stars keeps on decreasing as we move from centre to surface. Our given analysis can be applied to the interdisciplinary modeling, environmental and biological systems which may quite often involve complicated forms of linear or non-linear differential equation.

## References

1. R.H. Fowler and Grid properties of E-solutions as  $\xi \rightarrow 0$ , *MNRAS*, 91, 63-71 (1930).
2. E. Hopf, On solutions of Lane-Emden Equation for  $1 < n \leq 3$ , *MNRAS*, 91, 635-662 (1932).
3. E. A. Milne, Structure of self-gravitating polytropic sphere, *Hdb D ap*, 3, 65-71 (1930).
4. J.P. Sharma: Relativistic Stellar Structures and X-Ray Transients in Ni's Theory of Gravity, *Proc. Natn. Acad., Sc., India*, (2006).
5. Indira Bardoloi *et al.*, *Bull. Astr., India*, 32,1,2004.
6. J.P. Sharma, *Proc; Math. Soc., BHU*, 16, (2000).
7. J.P. Sharma, Thesis of D.Sc., Univ. of Gorakhpur (U.P.), India (1981).
8. J.P. Sharma, Thesis of D.Phil., Univ. of Allahabad (U.P.), (1970).
9. E.R. Harrison and R.G. Lake, *Ap. J.*, 171, 323, (1972).
10. J. Ostriker, *Ap. J.*, 140, 1056, 1964; also. *J. Suppl.*, 11, 167 (1965).
11. Shambhunath Srivastava and J.P. Sharma, *Experientia*, 24, 764 (1968).
12. D. Menzel, P.L. Bhatnagar and H.K. Sen, *Stellar Interiors*, Chapman and Hall, London, Chap. 10 (1963).
13. S. Chandrasekhar, *An Introduction to the Study Stellar structure*, Dover Publications, Inc., N.Y. (1939).
14. J. Homer Lane, *American Journal of Science*, 2dser, 50, 57,1869.
15. J.P. Sharma, Limiting density in white dwarfs, *Indian J Phys*, 48, 66-70 (1974).
16. R.A. Lyttleton, On the origin of mountains, *Proc Roy Soc*, 275, 1-11 (1963).
17. J.P. Sharma, Planetary structures in general relativity, *PAGEOPH*, 97, 14-24 (1972).
18. J.P. Sharma, On the maximum value of the mass of star, *Trans New York Acad Sc*, 34, 691-694 (1972).
19. Sanish Thomas, and J.P. Sharma, General relativistic Neutron stars, *Pramana* (2009).
20. J.P. Sharma, D.P. Jayapandian and Sanish Thomas, Very massive stellar models in Ni's theory of gravity, *Proc Nat Acad Sc India*, 79, 411-416 (2009).
21. J.P. Sharma, Relativistic stellar structures and X-ray Transients in Ni's theory of gravity. *Proc Nat Acad Sc. India*, 79, 29-34 (2009).
22. J.P. Sharma, Thermodynamically equilibrium of star cluster, *Proc Nat Acad Sc India*, 76, 305-307 (2006).