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website:- www.ultrascientist.org**A brief review of the Cuts in Integer Programming Problem**

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Abstract

The paper presents an overview of various cuts which are used in obtaining the integer solution to an Integer Programming Problem. The decision variables of any problem that must assume non-fractional values may be categorized as an integer programming.

The Cut is a device for reducing the feasible region. Whereas the feasible region is a set of feasible solution. Numerous mathematicians and researchers developed various Cuts for different Integer programming problem from time to time. An optimal solution is achieved in a minimum number of phases after using a developed Cut. Integer programming problems are widely used in many applications like Production planning, Scheduling, Telecommunications networks, Cellular network, etc.

Key words: Integer Programming, Linear Programming, Cuts, feasible region and optimal solution.

Introduction

Integer Programming Problem (IPP) :

The IPP is an optimization problem where each and every variable is needed to be an integer which is known as a pure integer programming problem. In case, a single variable is found to be non-integer, the problem will be known as a mixed integer programming problem. If we find that the integer variables are limited to be 0 or 1, then this type of problems will be referred to "pure (or mixed) 0-1 programming problems" or "pure

(or mixed) binary integer programming problems".

There are two main following grounds for using integer variables:

- The integer variables represent quantities that can only be integer. For example, it is not possible to build 5.7 motorcycle.
- The integer variables represent decisions and so should only take on the value 0 or 1.

Integer programming is not a new topic of Operations Research, but until the applications of operations research became recognized in the late 1940's and early 1950's, most of the problems tackled were primarily of

a pure scientific nature.

Round the non-integer optimum to the closet feasible integers is a common approach to solve the non-integer problem. For example, if the non-integer optimum indicates that the 'number' of machines required is 7.15, this can be approximated by (rounded to) 7. There is no assurance, however, that the rounded answer will always satisfy the constraints. This is always true if the original problem is linear with some equality constraints. From the concept of linear programming, rounded solution cannot be feasible, since it would imply that the same basis (with all its non-basic variables at zero level) can produce two different results.

Mathematical Model of ILPP :

The model of the Mixed IPP can be written as:
(MIP) maximize $z=cx + dy$

S.t,

$$\begin{aligned} Ax + Ey &\leq b \\ x &\geq 0, x \text{ integer-valued} \\ y &\geq 0 \end{aligned} \quad (1)$$

Where; $A = m \times n$ matrix; $E = m \times p$ matrix; $c=1 \times n$ vector; $d=1 \times p$ vector; $b=m \times 1$ vector; $x = n \times 1$ vector of integer variables and $y=p \times 1$ vector of continuous variables.

Where; p is the subset of n .

Similarly, the model of pure IPP as:

(ILP) maximum $z = cx$

Subject to,

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \text{ and integer} \end{aligned} \quad (2)$$

The feasible region associated with an integer program is always a subclass of the feasible region allied with its LP relaxation. Thus, for a maximization problem, the optimal value of objective function to the integer program will always be less than or equal to the optimal objective value of the LP relaxation. That is, for a maximization integer program, the LP relaxation provides an upper bound for the optimal objective value. Similarly the LP relaxation provides a lower bound in the case of a minimization integer program.

Approaches of IPP :

Approaches of IPP are given as :

- a. Search approaches
 - b. Cutting approaches
- a. *Search Approaches :*

The search approaches mainly comprise of 'implicit enumeration methods' and 'branch & bound' methods. The first type is mostly suited for the zero-one problem and may actually be considered as a special case of 'branch & bound' methods.

Land & Doig²⁶ explained the classical enumeration algorithm and its variations which have appeared in the literature. Little, Murty, Sweeney, and Karel²⁸ developed method for travelling salesman problem; which as mentioned in Balinski⁵. During 1965 'Dakin'¹³ suggested a variation of the Land and Doig algorithm.

Beale and Small^{9,10} described the MIP by a branch-and-bound technique. Driebeek¹⁷ suggested the linear programming post optimization procedures. Later, Toomlin^{34,35} described branch-and-bound approaches for integer programming. The research papers of Balinski⁵, Balinski & Spielberg⁶, Geoffrion and Marsten¹⁸, and Lawler and Wood²⁷ are survey articles which, among other things, contain an exposition of Land & Doig²⁶ method. A general description of branch and bound may be found in Again¹, in Balas²⁻⁴, in Mitten³¹. A procedure which specializes the Land and Doig approach to the zero-one case and incorporates the work of Driebeek¹⁷ and also the algorithm branching strategies proposed by Spielberg³² is described in Davis, Kendrick, and Weitzman¹⁶. Also Tobias Achterberg³⁷ presented Conflict analysis in mixed integer programming.

b. Cutting Methods :

The cutting techniques are established mainly for the (mixed or pure) integer linear programming. The idea then is to add specially developed secondary restrictions that are despoiled by the present non-integer solution but never by any feasible (integer) point. The successive application of such a procedure should eventually result in a new (convex) solution space with its optimum extreme point properly satisfying the integrality condition. The name "cutting" methods is suggested by the fact that the secondary constraints "cut" off infeasible parts of the continuous solution space.

A. Bockmayr and F. Eisenbrand³⁸ presented a research on combining logic and optimization in cutting plane theory.

The different types of cuts are as follows:

Dantzig Cut :

The early works of Dantzig *et al.*¹⁵ and Markowitz and Manne²⁹ directed the attention of researchers to the importance for solving ILP, but ‘Dantzig’¹⁴ was the first to propose a cut for solving such problems. His idea is first to solve the linear program ignoring the integer conditions. If the resulting basic solution is non-integer, then a new set of values for the non-basic variables must be secured. The form of the cut is

$$\sum_{j \in NBI} x_j \geq 1, \quad x_j \geq 0; j \in NBI \text{ (non-basic integer variables)}$$

This is completed by understanding that the sum of the non-basic integer variables at least equal to 1. By augmenting this cut to the current tableau, feasibility can be restored by applying the ‘dual simplex method’. Bowman, V.J., & Nemhauser¹² stated that the indicated cut represents a necessary condition for integrality and indeed it was utilized to solve some problems successfully, there is no guarantee that the successive application of the cut will produce the integer solution in a minimum steps.

Dual Fractional Cut :

This cut was developed by Gomory, R.E.²¹ and it is derived as:
Suppose source-row is

$$x_k = a_{k0} + \sum_{j \in NB} a_{kj} (-x_j)$$

The form of the cut is

$$S = -f_{k0} + \sum_{j \in NB} f_{kj} (-x_j)$$

$j \in NBI$ (Non-basic variable)

Where, $S \geq 0$ is a non-negative slack variables

$$\text{And } f_{k0} = a_{k0} - [a_{k0}]$$

$$f_{kj} = a_{kj} - [a_{kj}],$$

Where $j=(0, 1, 2, \dots, n)$; where, $[t]$ is the largest integer $< t$.

Gomory’s f-cut should be augmented to the simplex tableau from which it is derived. Since all $x_j=0$, $j \in NB$, it follows that $S = -f_{k0}$ which is feasible. Thus by applying the ‘dual simplex method’ a portion of the solution is cut off. If the resulting optimal solution is an integer, the procedure finishes.

Otherwise, a different cut is constructed from the new simplex tableau and the process is repeated. If it is impossible to recover feasibility after the cut is applied, this immediately means that the original problem has no feasible integer solution. The constraint is called the fractional cut (or f-cut) because all the coefficients of f_{k0} and f_{kj} are fractions.

All Integer Cut :

The use of the f-cut has the basic disadvantages that it gives rise to severe machine round-off error. Gomory²² rectified this difficulty by developing a new type of cut known as All Integer cut, which consists of all integer coefficients with its pivot element necessarily equal to 1. The idea of the new cut is to start with a dual feasible tableau with all integer coefficients. Suppose the source row is

$$x_k = a_{k0} + \sum_{j \in NB} a_{kj} (-x_j)$$

The form of the cut is

$$S = \left[\frac{a_{kj}}{\lambda} \right] + \sum_{j \in NB} \left[\frac{a_{kj}}{\lambda} \right] (-x_j) \geq 0 \quad (\lambda \text{ -cut})$$

Where λ is obtained by the following steps:

1. k as the source row, let α_p be lexicographically least column among $a_{kj} < 0$ for $j \in NB$.
2. Suppose $u_p=1$, and for every $j \geq 0$ ($j \neq p$); $a_{kj} < 0$, suppose u_j be the biggest integer such that, $\left(\frac{1}{u_j} \right) \alpha_j > \alpha_p$
3. For each $a_{kj} < 0$ ($j \geq 1$), set $\lambda_j = \frac{-a_{kj}}{u_j}$, (λ is not necessarily an integer)
4. Set $\lambda = \text{maximum of } \lambda_j$. Note that $\lambda > \lambda_p = \frac{-a_{kp}}{u_p} \geq 1$, since $u_p = 1$ and $-a_{kp}$ is a positive integer.

Mixed Integer Cut :

The mixed integer cut is given by Gomory²³ and it is defined as m-cut. Suppose the source row is

$$x_k = a_{k0} + \sum_{j \in NB} a_{kj} (-x_j)$$

The form of the cut is

$$S = -f_{k0} + \sum_{j \in NB} g_{kj} (x_j) \geq 0 \quad (\text{m-cut})$$

Where,

$$g_{kj} = \begin{cases} a_{kj}, & \text{if } a_{kj} \geq 0 \text{ and } x_j (j = 1, 2, \dots, n) \text{ is a continuous variable} \\ \frac{f_{k0}}{f_{k0} - 1} a_{kj}, & \text{if } a_{kj} < 0 \text{ and } x_j (j = 1, 2, \dots, n) \text{ is a continuous variable} \\ f_{kj}, & \text{if } f_{kj} \leq f_{k0} \text{ and } x_j (j = 1, 2, \dots, n) \text{ is an integer variable} \\ \frac{f_{k0}}{1 - f_{k0}} (1 - f_{kj}), & \text{if } f_{kj} > f_{k0} \text{ and } x_j (j = 1, 2, \dots, n) \text{ is an integer variable} \end{cases}$$

and $\sum_{j \in NB} g_{kj} (x_j) \geq f_{k0}$

$f_{kj} = a_{kj} - [a_{kj}] (j = 0, 1, 2, \dots, n)$

where, [t] is the largest integer <t,

and $f_{k0} = a_{k0} - [a_{k0}]$

Observe that $0 < f_{k0} < 1$ and when the inequality is adjoined to the bottom of the tableau, primal infeasibility is introduced.

Primal Cut :

Ben-Israels and Charnes¹¹ were the first to suggest a primal cut for IPP along the same ideas of Gomory's all integer λ -cut.

The form of the cut is

The form of the cut is

$$S = \left[\frac{a_{q0}}{\lambda} \right] + \sum_{j \in NB} \left[\frac{a_{qj}}{\lambda} \right] (-x_j) \geq 0 \quad (\text{P-cut})$$

With $\lambda = a_{qp}$

P-cut associated with source row, q is obtained by

$$\theta_p = \frac{a_{q0}}{a_{qp}} = \min_{a_{ip} > 0} \left\{ \frac{a_{i0}}{a_{ip}} \right\}$$

where p is the pivot column obtained using $a_{0p} =$

$$\min_{a_{ip} < 0, a_j \geq 1}$$

If $a_{ip} < 0 (i = 1, \dots, n + m)$, the IPP has an unbounded solution.

Otherwise from a row i with $a_{ip} > 0$ that satisfies

$$\left[\frac{a_{i0}}{a_{ip}} \right] \leq \theta_p, \text{ produce a 'Gomory all integer cut' and}$$

set λ in the cut equal to a_{ip} .

Bound- Escalation Cut :

This cut was developed by Glover¹⁹. Suppose the source row is

$$\sum_{j \in +j} a_{ij} w_j + \sum_{j \in -j} a_{ij} w_j \geq b_i \geq 0$$

The form of the cut is

$$y_k = w_k + \sum_{j \in +j} m_{ij} w_j \geq 0, \quad (\text{b-cut})$$

The general idea of Glover's cut is to find the optimal integer solution by determining lower bounds on each dual variable in such a way as to satisfy a necessary integrality condition.

Martin's f-Cut

This cut was developed by Martin³⁰. Suppose the source row is

$$x_i = \beta_i^r - \sum_{j \in NB - \{K\}} a_{ij}^i x_j - a_{is_r}^r S_r$$

The form of the cut is

$$S_r = a - \sum_{j \in NB} b_j x_j \quad (\text{Martin's cut})$$

This is an interesting variation of the basic cutting plane technique was proposed by Martin.

Convexity Cut :

This cut is a general form of legitimate integer cut that was initially started under the name "hyper cylindrical" cut by Young³⁶. Balas³ developed the intersection cut for solving general integer linear problems His idea is to utilize the local property of the integer points enclosing the current optimum continuous solution to develop a legitimate integer cut Actually, this cut is determined to pass through the intersection points of the half-lines emanating from the current optimum (continuous) extreme point with a specially defined hyper sphere The sphere passes through all the vertices of the unit hyper cube enclosing the continuous extreme point Later, Glover²⁰ developed a general theory for constructing legitimate cuts, which extends Young's and Balas' ideas to a general class of integer programming.

The general form of the convexity cut is $\beta x > \beta_0$; developed by Balas E³.

Naz Cut :

This cut was developed by Bari, A. and Ahmad, Q.S.⁷ as.

$$c x \geq z^0$$

This cut is added to the problem after finding the solution to LP relaxation problem. This cut is derived by finding the minimum perpendicular distance from the integer point, which is inside the feasible region, to the objective surface passing through non integer solution. The cut is the hyper plane passing through this point and parallel to the objective function surface. The cut has been designed in such a way that the total number of integer solutions in the resulting feasible region is substantially reduced.

A-T Cut :

This cut was established by Bari, A. and Alam,

$$\text{Teg}^8 \text{ as : } \sum_{j=1}^n x_j = \sum_{\substack{k^0 \in S^0 \\ j=1}}^n \alpha_j^{k^0}$$

Where;

k^0 = component of x^* (non-integer) with value $x^{k^0} = \alpha_j^{k^0}$.

The nearest integer solution to x^k are $x_1^k = [\alpha_1^k]$ and $x_2^k = [\alpha_2^k] + 1 = \langle \alpha_k^* \rangle$

To find an integer optimum from this reduced region we have developed a new cut (A-T cut). This cut passes through the integer points obtained during NAZ cut and any other integer points inside the reduced region. We have also proved that the integer optimum lies on this cut.

Conclusion

The importance of integer optimization in solving the practical problems evolved as a result of the impressive developments in the area of operations research, particularly the subject of liner programming. It was then that both the researchers and practitioners recognized the need for solving programming models in which some or all of the decision variables are integers. Although several important problems in various areas of application were formulated as integer models, it was only in 1958 that Gomory developed

the first finite integer programming technique for solving liner integer problems. Since then, other specialized algorithms as well as specialized cuts have been and are still being developed.

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